# Ordered Sets and Lattices
# Application to Coalitional Networks[*]

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The origin of the lattice concept can be traced back to Boole's analysis of thought and his project to turn set theory and logic into algebra. At the end of the nineteenth century, Ernst Schroeder and Charles S. Pierce introduced the concept of lattice while investigating the axiomatics of Boolean algebra. Lattice theory gained much of its importance in the 1930's and was largely diffused by the Birkhoff's book *Lattice Theory* in the 1940's.

## 1 Orderings

The idea of order appears in many aspects of the everyday life, to such an extent that we take it for granted : faster, bigger, better, etc. We use all these notions are commonly from our tender age : who runs faster, who is the tallest, ... Order is not a property intrinsic to a single object, it involves comparison between pairs of object : Richard is taller than Patrick, Mars is further from the sun than Earth, etc. Before plunging into the mathematical formalism of the idea of order and its properties, I first present some very common example from everyday life.

**Example 1.** *Some examples of orders :*

- $0 < 1$ *and* $1 < 10^5$.
- *The planets in order of increasing distance from the sun are Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune (and Pluto).*
- *Two first cousins have a common grandfather.*
- *Neither of the sets* $\{1,2,4\}$ *and* $\{2,3,5\}$ *is a subset of the other, but* $\{1,2,3,4,5\}$ *contains both.*

### 1.1 Partial Orderings

We start with a set $X$ with typical elements $x, y, z$. Throughout this note, sets are denoted by capital Latin letters and elements are denoted by small Latin letters. As usual, $\in$ and $\subseteq$ are the "belongs to" and (weak) inclusion symbols respectively.

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**Definition 1.1** (n-ary relation)**.** *Given sets* $X_1, X_2, \ldots, X_n$*, the* **cartesian product** *is the set of ordered n-tuples*

$$X_1 \times X_2 \times \cdots \times X_n \equiv \{(x_1, x_2, \ldots, x_n) | x_i \in X_i \textit{ for every } i\ \}$$

*An n-ary relation* $R$ *on* $X_1 \times X_2 \times \cdots \times X_n$ *is* $R \in \mathscr{P}(X_1 \times X_2 \times \cdots \times X_n)$ *where* $\mathscr{P}(X)$ *is the power set of* $X$*. Otherwise stated,* $R \subseteq X_1 \times X_2 \times \cdots \times X_n$ *.*

*If* $n = 2$*, then* $R$ *is a* **binary relation***. A binary relation can be defined for a given set* $X$*, in such a case* $R \subseteq X \times X$*, and we write* $xRy$ *if* $(x,y) \in R$*.*

**Definition 1.2** (Partial Ordering)**.** *A* partial ordering *on a set* $X$ *is a binary relation* $R$ *on* $X$*, that is* $R \subseteq X \times X$*, that satisfies the following conditions :*

- $(x,x) \in R$ *for all* $x \in X$ ***(reflexivity)**;*
- $(x,y),(y,x) \in R$ *imply* $x = y$ ***(antisymmetry)** ; and*
- $(x,y),(y,z) \in R$ *imply* $(x,z) \in R$ ***(transitivity)**.*

*The pair* $(X,R)$ *is called a* ***partially ordered set*** *or* ***poset****. A poset is a set whose elements are ordered but not all pairs of elements are required to be comparable in R.*

Intuitively, a binary relation on a set $X$ is simply a rule that decides whether or not $xRy$ for any given $x,y \in X$. Hence, a binary relation is simply the set $\{(x,y)|xRy, x \in X, y \in X\}$. Note that the three formal conditions are not independent. We can see directly that Transitivity and Antisymmetry imply Reflexivity.[1] The poset is $P = (X,R)$ and we write $x \leq y$ for $(x,y) \in R$, so the binary relation is $\leq$ and the poset $P$ is the pair $(X, \leq)$, and if $x \leq y$ we say that $x$ and $y$ are comparable with respect to $\leq$. A set $X$ equipped with the order relation $\leq$ gives rise to a relation $<$ of **strict inequality**: $x < y$ in $X$ if and only if $x \leq y$ and $x \neq y$. We can also define a companion relation to $\leq$ that will appear to be useful in the sequel : the **covering relation** on a poset $P = (X, \leq)$, denoted $x <^* y$. We say that $y$ covers $x$ in $X$ or that $x$ is covered by $y$ in $X$ if $x < y$ and there is no $z$ in $X$ such that $x < z < y$. The relation $x \parallel y$ (read "$x$ and $y$ incomparable"), means that neither $x \leq y$ nor $y \leq x$ holds. Another obvious relation is $x > y$ meaning that $y < x$.

In a poset $P = (X, \leq)$, we define the *interval* $[x,y]$ to be the set

$$[x,y] = \{z \in X : x \leq z \leq y\}.$$

By transitivity, the interval $[x,y]$ is empty if $x \nleq y$. If $y$ covers $x$ then $[x,y] = \{x,y\}$.

A poset is **locally finite** if all intervals are finite.

**Example 2.** *The poset* $(\mathbb{N}, \subseteq)$ *of natural numbers is infinite, but locally finite. On the other hand, The poset* $(\mathbb{Q}, \subseteq)$ *of rational numbers is not locally finite.*

If a poset is *finite* and $x \leq y$, then there exists a finite sequence of covering relations $x = x_0 \leq^* x_1 \leq^* \cdots \leq^* x_n = y$. Thus in the finite case, the order relation determines and is determined by the covering relation. This is shown formally thanks to the following theorem :

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[1] We can write alternately the Antisymmetry as follows : $(x,y) \in R$ and $x \neq y$ imply $(y,x) \notin R$.

**Theorem 1.1.** *Let* $P = (X, \leq)$ *be a finite poset. Then* $x \leq y$ *if and only if* $x = y$ *or there exists a finite sequence of elements* $x_0, \dots, x_{n-1}$ *such that* $x_0 = x$, $x_{n-1} = y$, *and* $x_i \leq^* x_{i+1}$ *for* $i = 0, \dots, n-1$.

*Proof.* The proof is by induction on the number of elements in the sequences. If there exists such a sequence between $x$ and $y$, a trivial induction on $n$ yields $x \leq y$. Thus it is sufficient to prove that if $x < y$ then there exists such a sequence. Fix $x, y \in P$, $x < y$ and take all subsets $H \subseteq P$ such that $H$ is a chain (where all elements are comparable with regard to the partial ordering induced by the partial ordering $P$) such that $x$ is the smallest element of $H$ and $y$ is the largest element of $H$. Such subsets exist : take $\{x, y\}$ for example. Choose such an $H$ with the largest possible number of elements. Suppose that $m$ is the number of elements of $H$. Then $H = \{x_0, \dots, x_m\}$ and we can assume that $x_0 < x_1 < \cdots < x_{m-1}$. We claim that in $P$ we have $x = x_0 <^* x_1 <^* \cdots <^* x_{m-1}$. Indeed, $x_i < x_{i+1}$ by assumption. Thus if $x_i <^* x_{i+1}$ does not hold, then there exists a $x' \in P$ such that $x_i < x' < x_{i+1}$, and $H \bigcup \{x'\}$ will be a chain in $P$ with $m + 1$ elements between $x$ and $y$, contradicting the maximality of the number of elements of $H$. ☐

Here are some examples of posets frequently met.

**Example 3.** *The relation* $R$ *over the power set* $\mathscr{P}(X)$ *given by* $(A, B) \in R$ *if and only if* $A \subseteq B$ *is a partial ordering. By abuse of notation, some may say that "*$\subseteq$*" is a partial ordering, without mention of* $R$.

*Similarly, we can define a partial ordering* $R$ *on the set of integers* $\mathbb{Z}$ *with the natural ordering "*$\leq$*".*

**Definition 1.3** (Partition)**.** *A partition* $A$ *of set* $X$ *is a collection of subsets* $A_i$ *of* $X$ *that we denote* $\{A_i\}_i$ *with* $i$ *taking values on a finite index set* $I = \{1, \dots, n\}$, *such that*

- *for all* $i$ : $A_i \neq \emptyset$, $A_i \subseteq X$,
- $i \neq j$ *implies* $A_i \cap A_j = \emptyset$,
- $\bigcup_{i \in I} A_i = X$.

**Example 4.** *A partition* $A = \{A_i\}_i$ *is finer than a partition* $B = \{B_j\}_j$ *if* $A_i \cap B_j \neq \emptyset$ *implies* $A_i \subseteq B_j$ *for some* $i, j$. *The relation "finer than" defines a partial ordering.*[2]

**Example 5.** *Some covering relations :*

- *In the chain* $\mathbb{N}$, *we have* $n <^* m$ *if and only if* $m = n + 1$.
- *In* $\mathbb{R}$, *there is no pair* $x, y$ *such that* $x <^* y$.
- *In* $\mathscr{P}(X)$, *we have* $A <^* B$ *if and only if* $B = A \cup \{x\}$ *for some* $x \in X \setminus A$.

Two special types of posets will appear to be important for further investigation : *Total Orderings* and *Lattices*.

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[2] Sometimes the antonym concept "coarser than" is used in place of "finer than" but never in pure combinatorics or graph theory.

## 1.2 Total Orderings

**Definition 1.4** (Total Ordering)**.** *A Total Ordering is a poset satisfying* ***Trichotomy*** *: for any $x, y \in X$, $(x,y) \in R$ or $(y,x) \in R$.*

In any poset, two elements $x$ and $y$ are *comparable* if $(x,y) \in R$ or $(y,x) \in R$, or using the alternative notation, in a poset $P = (X, \leq)$, we have $x \leq y$ or $y \leq x$ for any $x, y \in X$. Thus, a total ordering is a poset in which any two elements are comparable. A total ordering is also called a *linear order* and a totally ordered set is called a **chain**. Many concepts depart from the notion of chain. We can equivalently see a chain as a subset $C$ of mutually comparable elements of a poset $P$. The **length** of a finite chain $C$ is $\#C - 1$, the number of edges in the Hasse diagram (see definition below) of that chain, regarded as a poset. The **height** of a poset is the maximum size of a chain in that poset. An **antichain** or *clutter* or *Sperner family* is a subset $C$ of pairwise incomparable elements of a poset $P$. The *width* of a poset is defined to be the largest antichain in that poset. We say that a chain or an antichain is *maximal* if is contained in no other chain or antichain respectively.

Posets of subsets by inclusion, or partitions by the *finer than* relation are examples of nonlinear orderings. We now present an important and useful theorem, that need the following definition :

**Definition 1.5** (Isomorphic posets)**.** *Two posets $P = (X,R)$ and $Q = (Y,S)$ are isomorphic if and only if there exists a bijection (one-to-one and onto) $f : X \to Y$ such that $(x_1, x_2) \in R \Leftrightarrow (f(x_1), f(x_2)) \in S$. The bijection $f$ is called the poset isomorphism.*

**Theorem 1.2.** *If $\leq$ is a linear ordering defined on the n-element set $X$, then there exists a one-to-one function $f$ from the set $I = \{1, 2, \ldots, n\}$ onto the set $X$ such that*

$$f(i) \leq f(j) \text{ if and only if } i \leq j.$$

*Proof.* We proceed by induction on the cardinality $k$ of the set $X$. If $k = 1$, the theorem is trivially true. Now assume that the theorem is true for $k = n - 1$.

Pick up an element $x_1 \in X$. If there exist some $y \in X$ such that $x_1 < y$, choose one such $y$ and denote it $x_2$. Continue the process such that we end up with a sequence; for each $x_{i-1}$, if there exist some $y$ such that $x_{i-1} < y$, then let $x_i$ be one such $y$. The elements of the list of $x$'s are distinct, since $x_{i-1} < x_i$ and by transitivity $x_i < x_{i+1}$ for each $i$.

Eventually we meet an element $x_j$ such that there is no $y$ with $x_j < y$. By linearity of $L : y \leq x_j$ for every $y$. Now construct the set $X' = X \setminus \{x_j\}$ and the set $I' = \{1, \ldots, n-1\}$. By induction, there exists a function $f : I' \to X'$ such that $f(r) \leq f(s)$ if and only if $r \leq s$. Now extend the function $f$ to $X$ by defining $f(n) = x_j$. Because $x_j > y$ for every $y \in X$, $f(r) \leq f(s)$ if and only if $r \leq s$.

□

This theorem is really important because it says that *any* (finite) set linearly ordered is *isomorphic* to a set of integers with their natural ordering.

## 1.3 Maximal Element

**Definition 1.6** (Maximal element)**.** *Let $P = (X, \leq)$ be a poset and let $Q \subseteq P$, an element $x \in Q$ is a* **maximal** *element of $Q$ if $x \leq y \in Q$ implies $x = y$.*

*We say that $x \in Q$ is the* **greatest** *(or* **maximum***) element of $Q$ if $x \leq y \in Q$ for every $y \in Q$. If the full poset $P$ has a maximum element, it is called the* **top element** *of $P$ and is usually denoted $1_P$ or $P^\top$.*

An element is thus maximal if nothing else is "above" it, in contrast to the requirement that the element should be above everything, so there may be more than one maximal element. *Minimal elements* are defined similarly. If the full poset $P$ has a minimum element, it is called the **bottom element** of $P$ and is usually denoted $0_P$ or $P^\perp$.

**Theorem 1.3.** *Any (non-empty) finite poset $P = (X, \leq)$ contains a maximal element.*

*Proof.* Choose any $x_1 \in P$. If $x_1$ is not maximal, there exists $x_2 \in P$ with $x_1 < x_2$ (that is $x_1 \leq x_2$ and $x_1 \neq x_2$). Continue this process until a maximal element is reached, or we reach an element previously encountered. In the second case, we are in a cycle; if $i < j$, then

$$x_i < x_{i+1} < \cdots < x_{j-1} < x_j,$$

with $x_i = x_j$ which violates transitivity. So eventually a maximal element will be found. □

Pay attention in the theorem of the finiteness condition imposed on the set $X$. The argument obviously fails in infinite poset. There is no maximal element for the ordered set of integers for example.

## 1.4 New posets from existing ones

Very often we will have to deal with new posets built from existing ones, or that are subposet of a poset. The simplest case is the following one : Let $P = (X, \leq)$ be an ordered set and let $Q$ be a subset of $P$. Then $Q$ inherits an order relation from $P$. Given $x, y \in Q$, $x \leq y$ if and only if $x \leq y$ in $P$. We say that $Q$ has the order **induced from** $P$. We say that $(Q, \leq)$ is a **subposet** of $P = (X, \leq)$.

A subposet $C$ is called a **chain** in $P$ if $C$ is a linearly ordered set. A chain $C$ is **maximal** if for all $z \notin C$, $C \cup \{z\}$ is not a chain, *i.e.* there exists an element $x \in C$ such that $x$ and $z$ are incomparable.

A subposet $A$ of $P$ is an **antichain** if it consists only of incomparable elements. An antichain is **maximal** if for all $x \in X$, $x$ is comparable to an element in $A$.

Given a poset $(X, \leq)$, we can build a new ordered poset $(X^\delta, \leq)$ called **the dual of** $(X, \leq)$ by defining $x \leq y$ to hold in $X^\delta$ if and only if $x \geq y$ holds in $X$. This new set is very convenient because to each statement that holds true in $X$ corresponds a statement that holds in $X^\delta$. For example, if in $X$ there exists a unique element covered by exactly one other element, then in $X^\delta$ there exists a unique element covering exactely one other element. In general, for a statement $\Phi$ for an ordered set, we obtain the dual statement $\Phi^\delta$ by replacing each occurrence of $\leq$ by $\geq$ and vice versa. This leads to the following very convenient principle, that can be used to prove two theorems in the price of one :

**Principle 1** (The Duality Principle)**.** *Given a statement $\Phi$ about ordered sets which is true in all ordered sets, then the dual statement $\Phi^\delta$ is true in all ordered sets.*

Another important of families of ordered sets obtained from existing ones are the **ideals** and **filters**. Let $P$ be a partially ordered set and $Q \subseteq P$. We say that $\downarrow Q$ is a **order ideal** or a **down-set** of $P$ generated by $Q$ if, when $x \in Q$, $y \in P$ and $y \leq x$, then $y \in Q$. Thus

$$\downarrow Q = \{y \in P | \exists x \in Q : x \leq y\}.$$

Dually, We say that $\uparrow Q$ is a **order filter** or a **up-set** of $P$ generated by $Q$ if, when $x \in Q$, $y \in P$ and $y \geq x$, then $y \in Q$. Thus,

$$\uparrow Q = \{y \in P | \exists x \in Q : y \leq x\}.$$

Note that when $Q$ is a filter, $P \setminus Q$ is an ideal and vice versa. The family of all down-sets of $P$ is denoted $\mathscr{O}(P)$ and is itself an ordered set under the inclusion relation.

When the subset $Q$ is a singleton $\{x\}$, then we write $\uparrow x$ ans $\downarrow x$ and we call them **principal up-set** and **principal down-set** of $X$ generated by the element $x$.

Other poset obtained from existing ones will appear below with the sum and product of posets.

## 1.5 Hasse Diagram

There is a convenient way to represent a poset by its *Hasse Diagram*. An element *y covers* the element $x$ if $x < y$ and there is no element $z$ satisfies $x < z < y$. We can draw the Hasse (or covering) diagram of a poset $P = (X, \leq)$ in an Euclidean plane, such that each vertex corresponds to a point of the poset, and for each covering pair $x < y$, the points representing $x$ and $y$ are joined by an edge and the point representing $x$ has a smaller vertical coordinate than $y$ (it is located "below"). Examples and figures about here

## 1.6 Bounds

Let $S$ be a subposet of $P = (X, \leq)$.

- An element $x \in P$ is an **upper bound** of $S$ if $y \leq x$ for all $y \in S$. Let

  $$S^u = \{x \in P | y \leq x \text{ for all } y \in S\}$$

  be the set of all upper bounds of $S$. We say that $x \in S^u$ is a **least upper bound** (or a **supremum**) of $S$ if $x \leq y$ for all upper bounds $y \in S^u$. We denote the supremum (if it exists) of $S$ by $\bigvee S$.

- An element $x \in P$ is a **lower bound** of $S$ if $x \leq y$ for all $y \in S$. Let

  $$S^l = \{x \in P | x \leq y \text{ for all } y \in S\}$$

  be the set of all lower bounds of $S$. We say that $x \in S^u$ is a **greatest lower bound** (or a **infimum**) of $S$ if $y \leq x$ for all lower bounds $y \in S^u$. We denote the supremum (if it exists) of $S$ by $\bigwedge S$.

Because each finite poset posses a maximal element, if two elements of a finite poset have a lower bound, they must have a greatest lower bound, and may be not unique.

We note $S^{ul}$ for $(S^u)^l$. It represents the set of all lower bounds of upper bounds of $S$. Note that in general $S^{ul} \neq S^{lu}$.

Let $x, y \in P$. We use the following notation : $x \wedge y$ and $x \vee y$ are respectively the greatest lower bound and the least upper bound of $x$ and $y$ in a poset. They are also called the *meet* and the *join* of $x$ and $y$.

We now turn on to an important characterization of a poset $P$ taking advantage of the definitions of isomorphisms and down-sets :

**Theorem 1.4.** *Each poset P is isomorphic to the dual poset of its principal down-sets under set inclusion. Let* $\varphi : P \rightarrow \mathscr{P}(P)$ *be defined by*

$$\varphi(x) = \downarrow x.$$

*Then* $\varphi$ *is an order isomorphism from P onto the set of all principal down-sets of P.*

*Proof.* First, $\varphi$ is a bijection to the principal down-sets:

$$\varphi(x) = \varphi(y) \Leftrightarrow \downarrow x = \downarrow y \Leftrightarrow x \leq y \text{ and } y \leq x \Leftrightarrow x = y.$$

To show that $\varphi$ is an order isomorphism, observe that if $x \leq y$, then also $\varphi(x) \subseteq \varphi(y)$. Also, since $x \in \downarrow x$, $\varphi(x) \subseteq \varphi(y)$ implies $x \leq y$. Therefore $x \leq y$ if and only if $\varphi(x) \subset \varphi(y)$ and the claim follows. ☐

By the duality principle, it is direct to get the following theorem :

**Theorem 1.5.** *Each poset P is isomorphic to the poset of its principal up-sets under inclusion.*

## 1.7 Graded Posets

Let $P$ be a poset. A function $r : P \rightarrow \mathbb{N}$ is a **rank function** of $P$ if it satisfies the following conditions :

1. if $x$ is minimal, then $r(x) = 0$
2. if $x <^* y$ then $r(y) = r(x) + 1$.

Every finite chain possesses a unique rank function. But not all posets possess a rank function. For instance, the set of rational numbers with its natural order does not, because of its infinite descending chain. Even finite posets may fail to have a rank function (for instance, the smallest modular lattice with 5 elements, see below).

**Definition 1.7.** *We say that a poset P is* **graded of rank** *n if every maximal chain in P has the same length n, i.e., each maximal chain has* $n + 1$ *elements.*

**Theorem 1.6.** *Each graded poset of rank n has a unique rank function*

*Proof.* Take two maximal chains $C$ and $C'$ in $P$ and elements $x_i \in C$ such that $r_C(x_i) = i$. Then also $r_{C'}(x'_i) = i$. If $C$ and $C'$ have a common element $x_i = x'_j$, then necessarily $i = j$, otherwise suppose $i < j$, then the chain $x_0 < x_1 < \cdots < x_i = x'_j < \cdots < x'_n$ is longer then $C$, contradicting the fact that $P$ is graded. Therefore the rank functions of the maximal chains are compatible with each other. Since $P$ is the union of its maximal chains, the claim follows. $\square$

## 1.8 Linear extension of a poset

## 1.9 The Mobius function of a poset

### 1.9.1 The incidence algebra of a poset

Let $P$ be a locally finite poset. Denote

$$I(P) = \{f : P \times P \to \mathbb{R} | f(x,y) = 0 \text{ if } x \nleq y\}$$

the set of all real-valued functions for which $f(x,y) = 0$ if $x \notin \downarrow y$.

The **sum** and **scalar product** in $I(P)$ are defined by

$$(f+g)(x,y) = f(x,y) + g(x,y)$$

$$(cf)(x,y) = c.f(x,y)$$

for $c \in \mathbb{R}$ and $f, g \in I(P)$.

The **multiplication** or **convolution** is defined by

$$fg(x,y) = \begin{cases} \sum_{z \in [x,y]} f(x,z)g(z,y) & \text{if } x \leq y \\ 0 & \text{if } x \nleq y \end{cases}$$

This product is well-defined, since $P$ is locally finite and hence the summation is over a finite interval.

The **incidence algebra** of the locally finite poset $P$ is the set $I(P)$ together with the operations $+$, scalar product and convolution. If $P$ is finite, we can label all the elements of $P$ by $x_1, \ldots, x_n$ where $x_i < x_j \Rightarrow i < j$. Then $I(P)$ is isomorphic to the algebra of all upper triangular matrices $M = (m_{ij})$ over $\mathbb{R}$ where $1 \leq i,\ j \leq n$ such that $m_{ij} = 0$ if $x_i \nleq x_j$. It follows that a function $f \in I(P)$ has an inverse if and only if $f(x,x) \neq 0$ for all $x \in P$. In that case, $f^{-1}(x,y)$ depends only on $[x,y]$.

We denote $\delta$ (the **delta function** or **Kronecker function**) in $I(P)$ and it is defined by

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

The $\delta$ function is the **identity** element of the incidence algebra $I(P)$ : $f\delta = f = \delta f$ for all $f \in I(P)$. A function $f$ has an inverse $f^{-1}$ in $I(P)$ if $ff^{-1} = \delta = f^{-1}f$.

### 1.9.2 The inversion formula

We review in this section some useful functions in $I(P)$.

The **zeta function** of the poset $P$ is the characteristic function of the poset $P$ and it is defined by

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

The $\zeta$ function has several useful interpretations. First we have

$$\zeta^2(x,y) = \sum_{x \leq z \leq y} 1 = |[x,y]|$$

and more generally, if $k \in \mathbb{N}$ then

$$\zeta^k(x,y) = \sum_{x = x_0 \leq \cdots \leq x_k = y} 1,$$

the number of chains of length $k$ from $x$ to $y$, with equalities $x_i = x_{i+1}$ allowed.

We also have

$$(\zeta - \delta)(x,y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x = y. \end{cases}$$

Thus, $(\zeta - \delta)^k$ is the number of chains of length $k$ from $x$ to $y$ in $P$ with no repetition.

Because $\zeta(x,x) \neq 0$ for every $x \in P$, the $\zeta$ function has an inverse, which is called the **Mobius function** of $P$, denoted $\mu$. If $x \not\leq y$ then $\mu(x,y) = 0$ and

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & -\sum_{x \leq z < y} \mu(x,z) \text{ if } x < y. \end{cases} \tag{1.1}$$

Note that the summation is over an half open interval. If $x < y$ then we have by (1.1)

$$\sum_{z \in [x,y]} \mu(x,z) = 0. \tag{1.2}$$

**Lemma 1.1.** *Let $P$ be a locally finite poset. The mobius function $\mu$ of $P$ is the inverse of the zeta function $\zeta$.*

*Proof.* If $x < y$, by (1.2), we have

$$(\mu\zeta)(x,y) = \sum_{z \in [x,y]} \mu(x,z)\zeta(z,y) = \sum_{z \in [x,y]} \mu(x,z).1 = 0.$$

We also have that $\mu(x,x)\zeta(x,x) = 1$, hence $\mu\zeta = \delta$, and $\zeta\mu = \delta$. So $\mu^{-1} = \zeta$.

□

Note also that we could have defined the mobius function without reference to the incidence algebra, inductively.

**Theorem 1.7** (**The Mobius inversion formula).** *Let $P$ be a locally finite poset having a bottom element* $0$*, and let $f,g : P \to \mathbb{R}$ be functions. Then*

$$g(x) = \sum_{z\in[0,x]} f(z) \tag{1.3}$$

*if and only if*

$$f(x) = \sum_{z\in[0,x]} g(z)\mu(z,x). \tag{1.4}$$

**Theorem 1.8** (**The dual Mobius inversion formula).** *Let $P$ be a locally finite poset having a top element* $1$*, and let $f,g : P \to \mathbb{R}$ be functions. Then*

$$g(x) = \sum_{z\in[x,1]} f(z) \tag{1.5}$$

*if and only if*

$$f(x) = \sum_{z\in[x,1]} \mu(x,z)g(z). \tag{1.6}$$

**Theorem 1.9.** *Let $P$ and $Q$ be locally finite posets. The mobius function $\mu_{P\times Q}$ of the direct product $P\times Q$ is the product of the mobius functions $\mu_P$ and $\mu_Q$ of $P$ and $Q$, that is,*

$$\mu_{P\times Q}((x_1,y_1),(x_2,y_2)) = \mu_P(x_1,y_1)\cdot\mu_Q(x_2,y_2).$$

*Proof.* Let $(x_1,y_1) \le (x_2,y_2)$ in $P\times Q$. We have

$$\begin{aligned} \sum_{(x_1,y_1)\le(x',y')\le(x_2,y_2)} \mu_P(x_1,x')\mu_Q(y',y_2) &= \left(\sum_{x_1\le x'\le x_2} \mu_P(x_1,x')\right)\left(\sum_{y_2\le y'\le y_2} \mu_P(x_1,x')\right) \\ &= \delta_{x_1x_2}\delta_{y_1y_2} = \delta_{(x_1,x_2)\cdot(x_2,y_2)}. \end{aligned}$$

□

**Example 6.** *Consider the poset $P = (2^X,\subseteq)$ for a finite set $X$. The mobius function for $P$ is*

$$\mu(Z,Y) = \begin{cases} (-1)^{|Y\setminus Z|} & \textit{if } Z\subseteq Y \\ 0 & \textit{otherwise.} \end{cases}$$

*The mobius inversion formula becomes the following statement. Let $f,g : 2^X \to \mathbb{R}$; then*

$$g(Y) = \sum_{Z\subseteq Y} f(Z), \textit{ for all } Y\subseteq X,$$

*if and only if*

$$f(Y) = \sum_{Z\subseteq Y} (-1)^{|Y\setminus Z|} g(Z), \textit{ for all } Y\subseteq X\ .$$

**Theorem 1.10.** *Let $P$ be a locally finite poset, and denote by $c_i(x,y)$ the number of chains $x = x_0 < x_1 < \cdots < x_i = y$ of length $i$. Then,*

$$\mu(x,y) = \sum_{i \geq 0}^{\infty} (-1)^i c_i(x,y).$$

*Thus if $P$ is a finite poset with top element $1$ and bottom element $0$, let $c_i$ be the number of chains $0 = x_0 < x_1 < \cdots < x_i = 1$ of length $i$ between $0$ and $1$. Then*

$$\mu_P(0,1) = c_0 - c_1 + c_2 - c_3 + \ldots$$

**Corollary 1.** *Let $x,y \in P$, for a locally finite poset $P$. Then,*

$$\mu(x,y) = \sum_C (-1)^{|C|},$$

*where the summation is over all chains $C$ from $x$ to $y$, and where $|C|$ denotes the length of the chain $C$.*

## 1.10 Lattices

We first define the following binary operations in a poset $P : \vee : P \times P \to P$ and $\wedge : P \times P \to P$ called respectively the **join** and the **meet**.

**Definition 1.8** (Lattice)**.** *A (finite) lattice $L$ is a poset in which each pair of elements $x,y$ has a unique greatest lower bound and a unique least upper bound, denoted respectively $x \wedge y$ and $x \vee y$.*

We also adopt the following notation : for a subset $A = \{x_1, x_2, \ldots, x_n\}$ of a lattice $L$, we write $\bigvee A = x_1 \vee x_2 \vee \cdots \vee x_n$ and similarly for $\bigwedge A$. They are respecctively the least upper bound and greatest lower bound of $A$.

Consequently, a lattice has a unique minimal and a unique maximal element, that we denote respectivley 0 and 1. We have $0 \leq x$ and $1 > x$ for any element $x$.[3]

**Remark 1.** *Let $P$ be a non-empty poset. If $x \leq y$ then $x \vee y = y$ and $x \wedge y = x$. Hence to show that $X$ is a lattice, it suffices to show that $x \vee y$ and $x \wedge y$ exist in $X$ for all non-comparable pairs $x,y \in X$.*

**Remark 2.** *Let $P$ be an ordered set. If $x,y \in P$ and $x \leq y$, then $\{x,y\}^u = \uparrow y$ and $\{x,y\}^l = \uparrow x$. Since the least element of $\uparrow y$ is $y$ and the greatest element of $\downarrow x$ is $x$, we have $x \vee y = y$ and $x \wedge y = x$ whenever $x \leq y$. And since $\leq$ is reflexive, $x \vee x = x$ and $x \wedge x = x$.*

**Example 7.** *Any linear (totally) ordered set is a lattice : if $x \leq y$, then $x \wedge y = x$ and $x \vee y = y$.*

**Example 8.** *The power-set lattice $\mathscr{P}(X)$, whose elements are the subsets of a set $X$, ordered by inclusion. We have $x \wedge y = x \cap y$ and $x \vee y = x \cup y$.*

**Definition 1.9.** *A lattice $L$ is* **complete** *if both $\bigwedge A$ and $\bigvee A$ exist for all subset $A \subseteq L$. Thus if a lattice $L$ is complete, it does have a top and a bottom element:*

$$\bigvee L = 1 \text{ and } \bigwedge L = 0.$$

---

[3] Let 0 be a minimal element and $x$ be any element. If $z$ is the join of 0 and $x$, then $z \leq 0$, so $z = 0$ by minimality, whence $0 \leq x$. If $x$ is also a minimal element, then $x \leq 0$, hence $x = 0$ by antisymmetry.

## 1.11 Algebraic definition of a lattice

## 1.12 Products and dimension

If $(X,R)$ is not a linear order, we cannot be sure to be able to compare two elements $x$ and $y$. Suppose we want to compare some objects on the basis of different numeric attributes. If the element $x$ is better than $y$ on all these attributes, we are ready to classify them as $x$ beats $y$. But if it appears that $x$ is better for some attributes and $y$ is better on others, then, depending on the weight we attach to the attributes, we might come to different conclusions about their ordering, and we would be safer to say that $x$ and $y$ are not comparable. On this setting, where we have various numeric attributes to compare the elements of a set, we define :

**Definition 1.10** (Cartesian Product)**.** *$(X_1,\leq_1),\ldots,(X_n,\leq_n)$ are posets. The direct product or the iterated Cartesian product of these posets is the poset $(X,\leq)$ where*

$$X = X_1 \times \cdots \times X_n = \{(x_1,\ldots,x_n) : x_1 \in X_1,\ldots,x_n \in X_n\},$$

*and*

$$(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n) \text{ if and only if } x_i \leq y_i \text{ for } i = 1,\ldots,n.$$

We can show easily that $(X,\leq)$ is indeed a poset. Moreover, a direct product of lattices is a lattice, with meet and join defined by

$$\begin{aligned}(x_1,\ldots,x_n)\wedge(y_1,\ldots,y_n) &= (x_1\wedge y_1,\ldots,x_n\wedge y_n),\\ (x_1,\ldots,x_n)\vee(y_1,\ldots,y_n) &= (x_1\vee y_1,\ldots,x_n\vee y_n),\end{aligned}$$

and $0 = (0_1,\ldots,0_n)$, $1 = (1_1,\ldots,1_n)$.

**Proposition 1.1.** *If $|X| = n$, then the power-set lattice $\mathscr{P}(X)$ is the direct product of $n$ copies of the two-element lattice $\{0,1\}$.*

*Proof.* Let $X = \{x_1,\ldots,x_n\}$. Any $Y \in \mathscr{P}(X)$ can be identified by its characteristic function $(e_1,\ldots,e_n)$ where $e_i = 1$ if $x_i \in Y$, $e_i = 0$ otherwise. The characteristic function is a bijection between $\mathscr{P}(X)$ and $\{0,1\}^n$. Now, if $Y$ and $Z$ of $\mathscr{P}(X)$ have characteristic functions $(e_1,\ldots,e_n)$ and $(f_1,\ldots,f_n)$ respectively, then

$$\begin{aligned}Y \subseteq Z \quad &\Leftrightarrow \quad (\forall i)(x_i \in Y \Rightarrow x_i \in Z)\\ &\Leftrightarrow \quad (\forall i)(e_i = 1 \Rightarrow f_i = 1)\\ &\Leftrightarrow \quad (\forall i)(e_i \leq f_i),\end{aligned}$$

so the map is an isomorphism. ☐

Cartesian product is not the only way to relate posets. We present now two other relationship that can occur between posets :

**Definition 1.11** (Disjoint Union)**.** *The disjoint union of two posets $P = (X,R)$ and $P' = (X',R')$ with $X \cap X' = \emptyset$ is the poset $(X \cup X', R \cup R')$ that we denote $P + P'$.*

**Definition 1.12** (Linear Sum)**.** *The linear sum of two posets $P = (X,R)$ and $P' = (X',R')$ with $X \cap X' = \emptyset$ is the poset $(X \cup X', R \cup R' \cup (X \times X'))$, denoted $P \oplus P'$. This has the effect to put all of $P$ below the poset $P'$.*

## 1.13 Distributive and modular lattices

**Definition 1.13.** *A lattice $L$ is* **distributive** *if for all $x,y,z \in L$,*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

*and dually*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Note that for any lattice $L$, $x \wedge (y \vee z) \geq (x \wedge y) \vee (y \wedge z)$, for all $x,y,z \in L$. Hence, any lattice is half way to being distributive.

**Definition 1.14.** *A lattice $L$ is* **modular** *if for all $x,y,z \in L$,*

$$x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)$$

Note that for any lattice $L$, $x,y,z \in L$ with $x \leq y$, we have $x \vee (y \wedge z) \leq y \wedge (x \vee z)$. Because $x \leq y$ then $x \vee y = y$ and thus $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) = y \wedge (x \vee z)$. All distributive lattices are modular, but the converse is not necessarily true.

**Theorem 1.11.** *Let $L$ be a modular lattice and $x,y \in L$, then $[x \wedge y, x]$ is isomorphic to $[y, x \vee y]$.*

**Theorem 1.12.** *If $L$ is a modular lattice and $x,y \in L$ then, for the cover relation $\leq^*$ we have :*

$$x \wedge y \leq^* x \Leftrightarrow y \leq^* x \vee y.$$

**Definition 1.15.** *The* **cover chain** *of a lattice $L$ is a sequence $x = x_0 \leq^* x_1 \leq^* \cdots \leq^* x_n = y$ from $x$ to $y$ of length $n$.*

**Theorem 1.13** (Jordan-Dedekind)**.** *Let $L$ be a modular lattice. Then any two cover chains from $x$ to $y$ have the same length.*

## 1.14 Geometric lattices

In any modular lattice, the intervals $[x \wedge y, x]$ and $[y, x \vee y]$ are isomorphic. Now, we say that a lattice $L$ is **semimodular** if

$$x \wedge y \leq^* x \Rightarrow y \leq^* x \vee y,$$

for any element $x,y \in L$.

If a lattice $L$ is endowed with a bottom element 0, then an element $a \in L$ is an **atom** if $0 \leq^* a$. Denote $\mathscr{A}(L)$ the set of atoms of a lattice $L$. We say that a lattice $L$ is

- **atomic** if for all $x \in L$ such that $x \neq 0$, there exists an element $a \in \mathscr{A}(L)$ such that $a \leq^* x$.
- a **point lattice** if the atoms generate $L$, that is for all $x \in L$ with $x \neq 0$,

$$x = \bigvee \{a \in \mathscr{A}(L) | a \leq x\}.$$

- a **geometric lattice** if it is a semimodular point lattice that satisfies the finite chain condition.[4]

The subset lattice $2^X$ of a set $X$ is a point lattice. The atoms are the singletons $\{x\}$, and every subset $A \subseteq X$ is the join $\bigvee_{x \in A}\{x\}$. Moreover in $2^X$, we have that $A \leq^* B$ if and only if $|A \setminus B| = 1$ and $A \cap B = A \wedge B \leq^* A$ implies that $B \leq^* A \vee B = A \cup B$. Thus the lattice $2^X$ is semimodular. Every finite lattice satisfies the finite chain condition and thus for every finite set $X$, $2^X$ is geometric.

**Lemma 1.2.** *Let $L$ be a geometric lattice. If $a \in \mathscr{A}(L)$ and $x \in L$ are such that $a \nleq x$ then $x \leq^* x \vee a$.*

*Proof.* Since $a \nleq x$, we have $x \wedge a \leq a$ and since $a \in \mathscr{A}(L)$, it follows that $x \wedge a = 0 \leq^* a$. From the definition of geometric lattice we obtain that $x \leq^* x \vee a$ and the claim follows. $\square$

**Theorem 1.14** (Exchange rule)**.** *Let $L$ be a geometric lattice, and $a, b, \in \mathscr{A}(L)$ and $x \in L$. Then,*

$$a \nleq x \text{ and } a \leq x \vee b \Rightarrow b \leq x \vee a.$$

**Proposition 1.2.** *A locally finite lattice is geometric if and only if for all $a, b$,*

$$a \leq^* b \Leftrightarrow \exists \text{ a point } p \text{ with } p \nleq a, b = a \vee p.$$

**Definition 1.16.** *A lattice is called* **complemented** *if it has a* 0 *and* 1 *and for each element $a$ there exists a complement $a'$ such that $a' \wedge a = 0$ and $a' \vee a = 1$. The lattice is* **relatively complemented** *if every interval is complemented.*

## 2 Coalitional networks

We adopt the usual notation used in the context of networks. We add the following regarding coalitions :

*Partitions*

The players are forming coalitions that partition $N$. Denote $\mathscr{B}$ the set of all possible partitions or *coalitional structures* and $\#\mathscr{B}$ the cardinality of $\mathscr{B}$.
$P' \in \mathscr{B}$ is a *subpartition* of $P \in \mathscr{B}$ if $P'$ is finer in the interior sense than $P$ ($P' \succ P$):

$$\forall i, \exists j : P'_i \subseteq P_j$$

Given this property, if a partition $P'$ is finer in the interior sense than a partition $P$, we have $P'_i \cap P_j \neq \emptyset \Rightarrow P'_i \subseteq P_j$. We also say that $P$ is *coarser than* or is a *coarsening of* $P'$.

*Coalitional Network*

A coalitional network $(g, P)$ consists of a network $g \in \mathbb{G}$ and a coalition structure $P \in \mathscr{B}$

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[4] A poset $P$ satisfies the finite chain condition if every chain in $P$ is finite.

*Partition value functions*

A *partition value function* is a function $v : \mathbb{G} \times \mathscr{B} \to \mathbb{R}$. The set of all possible partition value functions is denoted $\mathscr{V}$.

Note that a partition value function is obviously more general than a value function defined on the set of possible networks, because it takes into consideration the possible coalitional structures, and at the same time the partition value function is also more general than a simple game whose domain is the power set of $N$. By taking into account partitions, we allow for externalities between coalitions.

## 2.1 Lattices in the context of coalitional networks

We already know that the set of subnetworks $g'$ of a network $g$, that we denote by its power-set $\mathscr{P}(g)$, forms a lattice ordered by graph-inclusion.

Now, we contemplate the partially ordered set of all partitions of a set $X$. The greatest lower bound of two partitions $P = \{P_i\}_i$ and $Q = \{Q_j\}_j$ of $X$ is easy to describe. Since each element $x$ of $X$ is in exactly one of the $P_i$'s and one of the $Q_j$'s, each $x$ is in an intersection $P_i \cap Q_j$, for some $i, j$. Now, each $P_i \cap Q_j$ is either empty or has no element in common with any other set $P_r \cap Q_s$. Thus the nonempty sets $P_i \cap Q_j$ form a partition of $X$. In the ordering "finer than", each set is a subset of one of the $P_i$'s and subset of one of the $Q_j$'s. Thus each set is a subset of $P_i \cap Q_j$ and consists in a greatest lower bound for $P$ and $Q$. From this, it follows that the partitions of $X$ form a lattice.

**Proposition 2.1.** *If $\mathscr{B}$ is the set of partitions of a set $X$ and if $R$ is the refinement ordering, then $(X,R)$ is a lattice.*

*Proof.* Any finite lattice has an element $t$ such that $t \geq x$ for all $x \in X$. But the set of all partitions has one element (a class) which is greater or equal to all partitions relative to the refinement ordering : the set $X$ itself. ☐

So now, we are able to state that a coalitional network $(g,P)$ is a direct product of lattices, namely the ordering by graph inclusion and the partition refinement :

$$(g,P) = g \times P = \{(g,P) : g \in \mathbb{G}, P \in \mathscr{B}\}$$

and

$$(g,P) \leq (g',P') \text{ if and only if } g \subset g' \text{ and } P \succ P'$$

In this setting, we can directly and easily define the following :

*A basis for value functions*

Let $v_{g,P}$ denote the value function that satisfies

$$v_{g,P}(g',P') = \begin{cases} 1 & \text{if } g \subset g' \text{ and } P \succ P' \\ 0 & \text{otherwise} \end{cases}$$

We call such $v_{g,P}$ a basic value function. Any $v \in \mathscr{V}$ can be uniquely written as a linear combination of basic value functions $v_{g,P}$'s :

$$v = \sum_{(g,P)} c_{g,P} v_{g,P}$$

for some unique collection of scalars $c_{g,P}$. Each $v$ can be seen as a vector in $\mathbb{R}^{(\#\mathbb{G}\times\#\mathscr{B})}$ noting that the set of possible $v_{g,P}$ are linearly independent and span $\mathbb{R}^{(\#\mathbb{G}\times\#\mathscr{B})}$. It is easy to see that there are $\#\mathbb{G}\times\#\mathscr{B}$ different possible basic value functions. We thus only have to show that the only solution to $\mathbf{0}=\sum_{(g,P)}c_{g,P}v_{g,P}$ is $c_{g,P}=0$ for every $g\in\mathbb{G}$ and $P\in\mathscr{B}$. Suppose not. Then there exists $(g,P)$ with minimal size.

*Efficiency*
A coalitional network $(g,P)$ is *efficient* relative to a partition value function $v$ if : $v(g,P)\geq v(g',P')$ for all $g'\in\mathbb{G}$ and all $P'\in\mathscr{B}$

*Monotonicity*
A partition value function $v$ is *monotonic* if for any for any $P\in\mathscr{B}$ and $g\in\mathbb{G}$ : $v(g',P')\geq v(g,P)$ when $P\succ P'$ and $g\subset g'$.
This condition simply says that the value function assigns greater value to elements higher in the lattice ordering.

*Monotonic covers*
If we adopt the point of view that a coalitional network is still in the process of construction, it will be useful to consider the value generated by the maximum over possible coalitional networks that could be formed. This is captured by the monotonic cover. Given a value function $v$, its *monotonic cover* $\hat{v}$ is defined by

$$\hat{v}(g,P)=\max_{(g',P')\leq(g,P)}v(g',P')$$

*Allocation rule*
Once the global value of a coalitional network is determined, an allocation rule specifies how this value will be allocated among the players.

An allocation rule is a function $Y:\mathbb{G}\times\mathscr{B}\times\mathscr{V}\rightarrow\mathbb{R}^n$ such that $\sum_i Y_i(g,P,v)=v(g,P)$ for all $g,P$ and $v$.

# 3 Flexible coalitional network and equal treatment

In this section we offer a characterization of allocations rules with the perspective that both the network and the coalitional structure can still be varied.

*Flexible-coalitional network rule*
An allocation rule $Y$ is a *flexible-coalitional network* rule if $Y(g,P,v)=Y(g^N,N,\hat{v})$ for all $v$ and efficient $(g,P)$ relative to $v$. The set of all subsets of size 2 is $g^N$ and $N$ is the grand coalition partition.

*Weak additivity*
An allocation rule $Y$ is *weakly additive* if for any monotonic $v$ and $v'$, and scalars $a,b\geq 0$,

$$Y(g^N,N,av+bv')=aY(g^N,N,v)+bY(g^N,N,v'),$$

and if $av - bv'$ is monotonic, then

$$Y(g^N, N, av - bv') = aY(g^N, N, v) - bY(g^N, N, v').$$

Because flexibility is allowed, what is important is to have separability on monotonic covers, as it is the real relevant information in these cases.

*Equal treatment of equal players*

An allocation rule $Y$ satisfies *equal treatment of vital players* if $v_{g,P}$ is a basic value function for some $g$ and $P$, then

$$Y_i(g, P, v_{g,P}) = \begin{cases} \frac{1}{n(g,?)} & \text{if } i \in N(?) \\ 0 & \text{otherwise} \end{cases}$$