Network Games as TU Cooperative Games :

The Core, the Shapley Value and Simple Network Games

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Abstract

In this article, we present and develop the interpretation of network game as a TU cooperative game where the players are the links of the original network game. Using this transposition, we propose the adaptation of two widely used concepts of cooperative game theory to the context of network games : the Core and the Shapley value. As illustration of the usefulness of the approach, we characterize the class of simple network games with non-empty Cores.

Keywords: Cooperative games, Network games, Networks, Shapley value, Core, Simple games

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1 Introduction

In this paper, we analyze networks from a cooperative point of view. In our framework players may form bilateral links. These links capture the way players can collaborate

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to achieve some productive value. Players are free to severe any link in which they are involved but the consent of two players is needed to build a link between them. Our goal is to analyze the stability of networks through payoff allocations of the total value generated by the networks. Since without any link no value is generated, we focus on allocations on a link-basis, i.e. players in the game receive payoffs through the payoff allocation to links they are involved in. We adapt in this context the notion of Core stable allocations : no subseteq of players can be "better off" in a new network with links only between members of that subseteq.

This paper exploits the novel approach that consists of transposing network games into TU coalitional games where the players are the links of the network games. This possible approach has already been mentionned in Slikker [15] but here, it is our intention to take full advantage of this transposition.

In a seminal paper, Myerson [10] led the way in adapting cooperative game theory structure to take into consideration information about the network connecting players. The role of the network in that setting is to determine which coalitions can function in the coalitional game, so that each network and characteristic function (determining the value of each possible coalition) induce a particular cooperative game. Myerson proposes then a method to allocate the value of the grand coalition : the Myerson value, which is the Shapley value of network-restricted coalitional games. In 1996, Jackson and Wolinsky [8] went further and considered a setting in which economic possibilities depend directly on the network structures connecting players. They call this setting *network games*. Jackson and Wolinsky study the relationship between the set of networks that are productively efficient and the set of networks that are (pairwise) stable. They also propose a generalization of the Myerson value in the context of network games. In this paper, we propose a different adaptation of the Shapley value in the context of network games. We suppose that allocations are link-based, that is ,any payoff distribution is realized through the links. The idea of assigning values to links first and then to players rather than directly to players was originated by Meessen [9] and Borm *et al.* [3]. It results in a variation of the Myerson value called the *Position value*. In our paper, the characterization of the Shapley value as link-based solution for network games follows closely the approach undertaken by Borm *et al.* [3], our set of properties characterizing the Shapley Value are very close to the axioms of Borm *et al*. In particular, their characterization is only valid on coalitional games restricted by undirected trees. We show that the transposition of network games into TU games renders the domain rich enough so that the translation of the original Shapley's axioms (that resembles the ones of Borm *et al.*) are also necessary and sufficient to characterize a unique solution. Note that the position value to a general communication game has been characterized by Slikker [14] and in Slikker [15] as the marginal value of a unique link-potential function.

In the next section we provide the definitions of TU games and Network games, as well as the translation of network games into TU games. In the third section, we adapt the cooperative game notions of Core and imputations to network games. We also characterize the set of network games with non-empty Cores. The section 4 is devoted to simple network games and their Cores. On the fifth section, we present and characterize the Shapley value as a link-based solution concept for network games. Last section concludes.

2 Cooperative games and network games

2.1 TU cooperative games

A *cooperative game with transferable utility* (TU game) is a pair (N, v) where $N = \{1, \ldots, n\}$ is the set of players and v is the *characteristic function* of the game, which assigns some real value to any subseteq S of N. We assume that $v(\emptyset) = 0$ always. Because each v has the family 2^N of all possible subseteqs of N as domain, each v can be seen as a vector in $\mathbb{R}^{2^n-1}$. In a game (N, v), a subseteq S of N is called a *coalition* and $v(S)$ is the *worth* the coalition S can guarantee to itself without the help of the other players in N. The set of all possible games on a player set N is denoted $\mathcal{V}^N$ whereas the set of all possible games is denoted $\mathcal{V} \equiv \bigcup_{n\in\mathbb{N}} \mathcal{V}^n$. In most cases we treat, the player set is fixed and v alone designates the *game*.

A *basis* for the set of games $\mathcal{V}^N$ consists of a list of games v_k, $k = 1 \ldots 2^n - 1$ such that any game in $\mathcal{V}^N$ can be written as a linear combination of the games in the basis and no game in the basis can be written as a linear combination of the remaining games in the basis.

For a throughout treatment of TU (and non TU) cooperative games, the reader is refereed to Peleg and Sudhölter [11].

2.2 Network games

A *network* g lists which pairs of players are linked to each other. A network is thus a list of *unordered* pairs of players $\{i, j\}$ or $\{j, i\}$ where $i \in N$ and $j \in N$, $i \neq j$. For simplicity, we use the shorthand notation ij to designate the pair $\{i, j\}$. We sometimes designate links by the letters l and k without mentioning the label of the players involved in the links. The complete network on N, in which any two players are linked is denoted g^N and is the set of all subseteqs of N of size 2. We denote the set of all possible networks on N by $G \equiv \{g : g \subseteq g^N\}$.

For later reference, we denote the network obtained by adding a link between i and j by $g + ij$ and we denote the network obtained by deleting the link ij from an existing network g by $g - ij$. The networks $g + l$ and $g - l$ are defined similarly, with l a link between two players.

The set of players who have at least one link in a network g is $N(g) \equiv \{i : \exists j \text{ s.t. } ij \in g\}$ and $n(g)$ is the cardinality of $N(g)$, i.e. the number of players involved in a network relationship by g.

Let $L_i(g)$ be the set of links of player i in the network $g : L_i(g) \equiv \{ij : ij \in g\}$. If $l_i(g)$ is the cardinality of $L_i(g)$ then $\sum_i \frac{1}{2} l_i(g) \equiv l(g)$ is the total number of links in g.

For any given coalition $S \subseteq N$ and network g, we denote by g^S the complete network among the players in S whereas $g_{|S}$ is the network found by deleting all links of g involving at least one player outside S, i.e. $g_{|S} \equiv \{ij : ij \in g \text{ and } i \in S, j \in S\}$.

A *component* is a maximally connected subnetwork g' of a network g such that between any two players i and j in g', there exists a sequence of links $\{i_1 i_2, i_2 i_3, \ldots, i_{k-1} i_k\}$ in g where $i_1 = i$ and $i_k = j$ and there is no sequence of links leading to a player outside g'.

A *network game* is a pair (N, r) where N is the set of players and r is a *value function*[1] that assigns a real value to each possible network on N. A value function keeps track of the total value generated by a given network structure (in the same vein a characteristic

[1] Slikker [15] calls such a function a *reward function*.

function specifies a worth for every coalition in TU games). Thus, $r(g)$ represents the worth generated by the players in N while organized through the network g. If there is no link between the players, $r(\emptyset) = 0$ where $\emptyset$ is the network without links, it implies that if there is no cooperation between the players then no worth is generated. The set of possible value functions on N is $\mathcal{R}^N$ and the set of all possible network games is $\mathcal{R} \equiv \bigcup_{n \in \mathbb{N}} \mathcal{R}^n$.

2.3 A basis for network games

As for the set of possible TU games, we can define a *basis* for the set of possible network games as a list of network games such that any network game can be expressed as a linear combination of the games in the basis whereas no game in the basis can be expressed as a linear combination of the remaining games in the basis.

Except when otherwise specified in the sequel, the player set N is considered as fixed. In such case, a network game is identified with its value function $r \in \mathcal{R}^N$.

Let r_g denote the value function defined by

$$r_g(g') = \begin{cases} 1 & \text{if } g \subseteq g' \\ 0 & \text{otherwise.} \end{cases}$$

For any $g \subseteq g^N$, we call such value function r_g a *basic value function*. The set of all possible basic value functions forms a basis for network games :

Proposition 2.1. *Let $r_g \in \mathcal{R}$ be as defined above. The set $\left\{r_g \in \mathcal{R} : g \subseteq g^N, g \neq \emptyset\right\}$ of all basic value functions forms a basis for $\mathcal{R}$.*

Proof. We first note that there exists $(2^{n(n-1)/2} - 1)$ different r_g's. If we view each value function $r \in \mathcal{R}$ as a vector in $\mathbb{R}^{2^{n(n-1)/2}-1}$, we only have to show that any two different r_g's are linearly independant to conclude that the set of r_g's forms a basis of $\mathcal{R}$.

Suppose there exist real numbers c_g's, $g \subseteq g^N$, $g \neq \emptyset$, such that $\sum_{\emptyset \neq g \subseteq g^N} c_g r_g(g') = 0$ for all $g' \subseteq g^N$. We prove by induction on $l(g) \equiv \sum_i \frac{1}{2} l_i(g)$ that $c_g = 0$ for all $g \subseteq g^N$.

1. If $l(g) = 1$ then the equivalence $r_{g'}(g) = 1$ if and only if $g = g'$ holds and hence

$$0 = \sum_{\emptyset \neq g' \subseteq g^N} c_{g'} r_{g'}(g) = c_g r_g(g) = c_g.$$

This completes the proof of the induction for $l(g) = 1$.

2. Now, let $2 \leq l(g) \leq n(n-1)/2$ and suppose that $c_{g'} = 0$ for all $g' \subseteq g^N$ with $1 \leq l(g') \leq l(g)$. For all $g' \subseteq g^N$ with $l(g') \geq l(g)$ we have $r_{g'}(g) = 1$ if and only if $g = g'$. It follows that

$$0 = \sum_{\emptyset \neq g' \subseteq g^N} c_{g'} r_{g'}(g) = \sum_{g'; l(g') \geq l(g)} c_{g'} r_{g'}(g) = c_g r_g(g) = c_g.$$

We conclude that $c_g = 0$ for all $g \subseteq g^N$, $g \neq \emptyset$.

This completes the inductive proof of the linear independence of the set of all basic value functions.

□

2.4 Allocation rules for network games

Thanks to a value function $r \in \mathcal{R}$, we know the total value generated by each possible network g on a player set N. How should we redistribute the fruit of the network collaboration among players ? This is described by an allocation rule :

Definition 2.1. *An allocation rule*[2] *is a function* $Y : G \times \mathcal{R} \to \mathbb{R}^n$ *such that* $\sum_i Y_i(g, r) = r(g)$ *for all* $r \in \mathcal{R}$ *and all* $g \subseteq g^N$.

Each $Y_i(g, r) \in \mathbb{R}$ is interpreted as the payoff player i receives by collaborating in the network g whose worth is given by $r(g)$. Note that the definition of the allocation rule forces the value of a given network g to be fully redistributed among the players. Jackson [6] calls this condition *balancedness*.

Because an allocation rule is a way to allocate the total value generated by a set of players collaborating in a two-by-two basis, the allocation rule has to take into account the marginal value of a player, that is, the value added to the network by this player's participation. Moreover the allocation rule should also take into account the bargaining power of a player : what would happen to the total productivity of a network if that player were to break one or several of her links ? Even if players control the links and decide which link to create or to severe, the value generated by any network is due to the presence

[2] or allocation *scheme* for Slikker [15].

or the absence of the possible links between players. Thus, in a sense, how the total value has to be redistributed may be done in terms of the links controlled by the players. This leads to the following definition :

Definition 2.2. *An allocation rule is linked-based if there exists a function* $\psi : G \times \mathcal{R} \to \mathbb{R}^{n(n-1)/2}$ *such that* $\sum_{ij \in g} \psi_{ij}(g, r) = r(g)$*, and*

$$Y_i(g, r) = \sum_{j \neq i} \frac{\psi_{ij}(g, r)}{2}.$$

This means that the total productivity of a network is indirectly allocated to the players, the value is first distributed to the links and then to the players controlling the links. Note that in this definition, we consider that both players in a given link are equally important to form and maintain the link. We assume that the value of the link is equally distributed among the two players.[3]

For more discussion and definitions about network games, the reader is refereed to Jackson [5] or Jackson [7].

2.5 Network games as TU cooperative games

So far we have defined two different types of games, one which assigns value to any *coalition* or subseteq of players and one which assigns value to any network of players. It is well known that the second type of games is a richer object than the first type of games. This is due to the fact that in TU games, collaboration is transitive : if a player i collaborates with a player j, and if player j collaborates with a player k, all three players belong to the same coalition. Thus, whether player i directly collaborates with player k or not has no impact on the value of their collaboration. In the context of networks, intransitivities are allowed. In a setting where players i and j collaborate, and so do players j and k but not i and k will generally generate a different productivity value than the structure in which all three players collaborate jointly. This can be directly seen in their respective domain of definition. For a given set of players N, TU games are defined on

[3] This is the same assumption as in Borm et al. Nevertheless, as suggested by Wooders (private communication), it could be interesting to explore some general rule of distribution of the value of the link, for example when the bargaining power of the players is different or if the players are selected randomly.

a 2^n-dimension domain and Network games are defined on a $2^{n(n-1)/2}$-dimension domain. Nevertheless, as Slikker [15] puts it, network games can be seen as TU-games : the value function r of the network game (N, r) can be seen as the characteristic function of the TU game (g^N, r). The trick is to consider the set of links as the new set of players and using the value function as the characteristic function of the new game. Slikker calls this TU game *the link game* associated with the network game (N, r). This operation mimics the building of an edge-graph[4] from a given graph where each edge of the original graph is a vertex in the new graph : players in the associated link game are the links of the network game under consideration.

Definition 2.3. *Let (N, r) a network game with value function r. The associated link game of (N, r) is a TU game (g^N, r) where the set of players is the set of possible links g^N and the characteristic function is the value function r of the network game.*

Example 1. *Let $N = \{1, 2, 3\}$ and $r \in \mathcal{R}^N$ be such that $r(g) = l(g)$ for any $g \subseteq g^N$. Then, the associated link game (g^N, r) is such that $r(S) = |S|$ for any $S \subseteq g^N$.*

Note that this procedure is not without loss of generality. Consider two identical networks defined on the same set of players. Suppose we add a player in the set of players of one of the networks, and we leave that player isolated (since he has no link, the structure of the network is not modified). The other network remains unchanged. Then the value function have to attribute the same value to both networks in order to be compatible with an associated link game. An example may be useful to illustrate this fact.

Example 2. *Let g be the one link network defined on $N = \{1, 2\}$ and g' be the network defined on $N' = \{1, 2, 3\}$ with a link between player 1 and 2, and player 3 isolated. Any value function $r \in \mathcal{R}$ such that $r(g) \neq r(g')$ has no associated link game (because both associated link games are different valued one player games).*

The class of value functions that admits an associated link game is reduced to the class of value functions such that the value that accrues to a network is by no mean influenced by the presence of isolated players. This restriction may be thought as a serious one, but it is nevertheless less restrictive than the condition of component additiveness of

[4]also called adjoint graph, derived graph, ... see Balakrishnan [1].

value functions, which implies the absence of influence of isolated players.[5] Component additiveness precludes externalities across components, that is, the value taken by a component does not depend on the way the other players are arranged. Compatibility with associated link game only precludes the value of the whole network to be influenced by an isolated player. Because we are interested in settings where productive collaborations occur by establishing connections between players, disregarding isolated players should not be considered as so harmful. This is explicitly stated in the zero-value of the empty network. Conveniently, this also means that the knowledge of $r \in \mathcal{R}^N$ is sufficient to recover the behavior of $r \in \mathcal{R}^{N-k}$, with $k = 1, \ldots, n-1$ by just isolating some players in N.

3 Imputations and the Core of network games

In this section we adapt some important solution concepts from TU games to the context of network games. Once we know the total productive value of the different possible networks, a natural question that arises is how to distribute the fruit of these productive collaborations among the players ? We have seen in section 2 how to answer this question thanks to an allocation rule, that defines an allocation $\mathbf{x} \in \mathbb{R}^{\mathbf{N}}$ for each possible value function r. We now turn on to different motivations that can be put forward. We define a vector whose components indicate how payoff is distributed, but we relax the assumptions made in the definition of an allocation rule and impose some other conditions. We first begin with some stability consideration.

Definition 3.1. *Let (N, r) be a network game. A vector $\mathbf{x} \in \mathbb{R}^{n(n-1)/2}$ is called an imputation if*

1. $\mathbf{x}$ *is link rational :*
$$x_l \geq r(l) \qquad \text{for all } l \in g^N,$$
2. $\mathbf{x}$ *is efficient :*
$$\sum_{l \in g^N} x_l = r(g^N).$$

[5] The way a component is defined implies that a player with no link is not considered as a component.

We denote the set of imputations of (N, r) by $I(r)$ and any element $\mathbf{x}$ of $I(r)$ is a payoff distribution of the total worth of the complete network g^N which gives each link $l \equiv ij$, $i \neq j$, a payoff x_{ij} which has at least the same value the link can guarantee to itself if it is the only link in the network. Note that the set of imputations for a network game with value function r is nonempty if and only if

$$r(g^N) \geq \sum_{l \in g^N} r(l).$$

The equivalent concept to link rationality in the context of (classical) TU games is the *individual rationality*, where each player should receive at least the payoff he would receive when he operates alone, i.e. as singleton. As the discussion in the previous section made clear, whenever a player operates alone in a network game, he is isolated and receives zero payoff. If we see the set of imputations as the collection of marginal values or reservation payoffs, it seems more natural to define an imputation on a link basis rather on a player basis where each player, as isolated node, has zero marginal value to the empty network. Of course, we could argue that the marginal value of a player in a network game to the empty network is the sum of the values of the possible links this player can make. Nevertheless, the making of a link needs the consent of a partner. This necessity of a partner renders the very notion of *marginal* a little bit flaw, because the partner will also generate some value. Hence, in the context of network games, we found it more natural to use the notion of link rationality rather than individual rationality.

What is the relation between the set of imputations and the set of allocation rules ? Let $\mathbf{x}$ be an imputation for a network game (N, r). Construct $Y_i = \sum_{j \neq i} \frac{x_{ij}}{2}$. We have no guarantee that $\sum_{i \in N} Y_i = r(g)$ for all $g \subseteq g^N$, thus Y may fail to be an allocation. Now let $Y_i(g, r)$ be a link-based allocation as described in definition 2.2. Construct $\mathbf{x} = \{x_{ij}\}_{ij \in g^N}$ be such that

$$x_{ij} = \frac{Y_i(g, r)}{l_i(g)} + \frac{Y_j(g, r)}{l_j(g)}.$$

But then, we have no guarantee that $x_{ij} \geq r(ij)$ for all $ij \in g^N$, and $\mathbf{x}$ is not an imputation. The reason why the set of imputations is different than the set of allocations is the local properties of imputations : an imputation only requires link rationality and distribution of the value of the complete network only. Whereas an allocation rule specifies how to distribute the value of any network, given the value taken by the network. Hence, we see that if $g = g^N$, then an imputation is a (link-based) allocation.

We now restrain the set of admissible imputations to payoff distributions that are *coalitionally* rational. Again, we have a network game (N, r) and a vector $\mathbf{x} \in \mathbb{R}^{n(n-1)/2}$. We say that a vector $\mathbf{x}$ is a *feasible* payoff distribution for a coalition S of players ($S \subseteq N$) if and only if for any network $g \subseteq g^N_{|S} \equiv g^S$, we have :

$$\sum_{l \in g} x_l \leq r(g).$$

Hence, the players in coalition S can collaborate together in a network g and distribute the worth $r(g)$ among their links as prescribed by the components of $\mathbf{x}$. A payoff distribution is simply *feasible* if it is feasible for the grand coalition N. We say that a coalition of players $S \subseteq N$ can improve on a payoff vector $\mathbf{x}$ if and only if $r(g) > \sum_{l \in g} x_l$ for any $g \subseteq g^S$. This last condition says that there exists an other vector $\mathbf{y} \in \mathbb{R}^{n(n-1)/2}$, $\mathbf{y} \neq \mathbf{x}$ such that $\mathbf{y}$ is feasible for S and the links of the players in S get a strictly higher payoff in $\mathbf{y}$. We say that the payoff distribution $\mathbf{y}$ *dominates* $\mathbf{x}$ or that the payoff distribution $\mathbf{x}$ is *dominated* by $\mathbf{y}$. The *Core* of a network game (N, r) is simply the set of feasible payoff distribution vectors that are undominated, that is there is no coalition for which there exists an alternative feasible payoff distribution that gives to every link a better payoff.

Definition 3.2. *Let (N, r) be a network game. The Core of the game (N, r) is the set*

$$C(r) \equiv \left\{ \mathbf{x} \in I(r) : \sum_{l \in g} x_l \geq r(g) \textit{ for all } g \subseteq g^N,\ g \neq \emptyset \right\}.$$

The Core is a very important stability concept for TU games. For network games, if a payoff distribution $\mathbf{x}$ is not in the Core of the game, then there exists a set of players that prefers to be linked under a network $g \neq g^N$ and receive $r(g)$. It is so because the total amount $\sum_{l \in g} x_l$ allocated to the players in g under $\mathbf{x}$ is smaller than the amount $r(g)$ which the players can obtain by forming the subnetwork. If the payoff distribution $\mathbf{x}$ is in the Core of the game, then no profitable reorganization of the network exists such that each link in which the player are involved receives a better payoff. As in TU games, the Core of a network game may be empty.

Example 3. *Let* $N = \{1,2,3\}$ *and* $r \in \mathcal{R}^N$: $r(\{12\}) = r(\{13\}) = r(\{23\}) = 0$, $r(\{12,23\}) = 4$, $r(\{12,13\}) = 3$, $r(\{13,23\} = 2$ *and* $r(g^N) = 4$. *Then the Core of* (N,r) *is empty, since we have* $2(x_{12} + x_{23} + x_{13}) \geq 9$ *which is incompatible with the efficiency requirement* $\sum_{l \in g^N} x_l = 4$.

By definition 2.2, if the Core of a game is not empty then a link-based allocation is in the Core of the game. The characterization of the class of network games with non-empty Cores is based on a characterization due to Bondareva [2] and Shapley [13] for the class of games in which the Core is nonempty. Their characterization relies on duality theory of linear programming. To understand this Core existence problem, we can restate the definition of the Core as follows : what is the minimum amount of total payoff that is necessary such that no subseteq of players can improve upon by altering the network ? In mathematics this question corresponds to the following linear programming problem :

The objective is to

$$\min_{\mathbf{x} \in \mathbb{R}^{n(n-1)/2}} \sum_{l \in g^N} x_l \tag{3.1}$$

with the linear constraints

$$\sum_{l \in g} x_l \geq r(g) \text{ for all } g \subseteq g^N,\ g \neq \emptyset. \tag{3.2}$$

We remind that we denote by G (or $G(N)$) the set of all possible networks among the players in N, that is

$$G \equiv \{g : g \subseteq g^N\}$$

and the cardinality of G is denoted $|G|$.

The dual of the linear program (3.1)-(3.2) is thus :

$$\max_{\lambda(g) \in \mathbb{R}_+^{|G|}} \sum_{g \subseteq g^N} \lambda(g) r(g) \tag{3.3}$$

subject to

$$\sum_{g \supset l} \lambda(g) = 1, \text{ for all } l \in g^N. \tag{3.4}$$

Before we provide the theorem stating Core non-emptiness, we provide some definitions of concepts derived from the dual program.

Let N be the finite set of players. A map $\lambda : G \to \mathbb{R}_+$ is called a *balanced map* if

$$\sum_{g \subseteq g^N} \lambda(g) e^g = e^{g^N}$$

where e^g is a characteristic vector for network g with components

$$e_l^g = 1 \text{ if } l \in g \text{ and } e_l^g = 0 \text{ if } l \in g^N \setminus g.$$

A collection B of networks is called *balanced* if there is a balanced map λ such that $B = \{g \in G : \lambda(g) > 0\}$.

A network game (N, r) with $r \in \mathcal{R}^N$ is *balanced* if for each balanced map $\lambda : G \to \mathbb{R}_+$, we have

$$\sum_{g \subseteq g^N} \lambda(g) r(g) \leq r(g^N).$$

Theorem 3.1. *Let (N, r) be a network game with $r \in \mathcal{R}^N$. Then the following two assertions are equivalent :*

(i) $C(r) \neq \emptyset$.

(ii) (N, r) *is a balanced network game.*

Proof. The non-emptiness of the Core in (i) is satisfied if and only if the linear program (3.1)-(3.2) is satisfied and the optimal value is $r(g^N)$. By the duality theorem (see *e.g.* Franklin [4], p.80), $r(g^N)$ is the optimal value of the dual program (3.3)-(3.4), as both programs are feasible. Hence, (N, r) has a nonempty Core if and only if

$$r(g^N) \geq \sum_{g \subseteq g^N} \lambda(g) r(g) \tag{3.5}$$

which states the equivalence between (i) and (ii). □

4 Simple network games

In this section we introduce a special class of network games : the *simple network games*. A network game is *simple* if any possible network takes the value of 1 or 0.[6] Formally :

[6]Sometimes in simple TU games, the condition of monotonicity is also added to the definition of simple game : supersets of any coalition with value 1 have a value of 1.

Definition 4.1. *Let* (N, r) *be a network game.* (N, r) *is a simple network game if and only if*

$$r(g) \in \{0, 1\}$$

for all $g \subseteq g^N$ *and*

$$r(g^N) = 1.$$

A simple network game describes a situation where some networks are *winning* and the other possible networks are *losing*, that is only certain configurations can achieve positive value (of 1) according to the value function of the network game. In such a case, a simple network game (N, r) can equally be described by its set of *winning networks* : $W(r) \equiv \{g \subseteq g^N : r(g) = 1\}$. The networks in G that are not a member of $W(r)$ take the value 0. The complete network always has the value 1.

Some players in a network game can play a special role in the determination of the value taken by networks. For example, the presence of certain players may be essential in a network to achieve some positive value. This kind of player is called a *veto player*.

Definition 4.2. *A player* i *is a veto player for a network game* (N, r) *if and only if*

$$r(g) = 0 \qquad \text{for any } g \subseteq g^{N \setminus \{i\}}.$$

This means that the cooperation of the veto player i is required to obtain profits, and any network without i achieves 0 value. We say that a network game is a *network game with veto control* if there is at least one veto player.

Accordingly, we may imagine some network games that can achieve some nonnegative value if and only if we observe the collaboration between two specific players, that is the presence of a given link is necessary. We call this kind of link an *essential link*.

Definition 4.3. *A link* l *is an essential link for a network game* (N, r) *if and only if*

$$r(g) = 0 \qquad \text{for any } g \subseteq g^N \setminus \{l\}.$$

If we want a network to be stable against reorganization of links in an alternative network, we have to chose a Core allocation. In the following theorem we show that the Core of a simple network game is nonempty if and only if the game has at least **two veto players**.

Theorem 4.1. *Let (N, r) be a simple network game. Then, the Core of (N, r) is nonempty if and only if there are least two veto players.*

Before proving this theorem, we show that the presence of at least two veto players is equivalent to the presence of an essential link.

Proposition 4.2. *A simple network game (N, r) has an essential link $l = ij$ if and only if player i and player j are veto players for (N, r).*

Proof. Suppose that player i is a veto player for the simple network game (N, r). By definition 4.2, $r(g) = 1$ if and only if $g \supset g^{N\setminus\{i\}}$. Suppose that player j is also a veto player. By definition 4.2 applied twice, $r(g) = 1$ if and only if $g \supset \left\{g^{N\setminus\{j\}} \cap g^{N\setminus\{i\}}\right\} = g^N \setminus \{ij\}$. Hence, the link ij is essential.

Now, suppose that the link ij is an essential link for the simple network game (N, r). Then by definition 4.3, $r(g) = 1$ if and only if $g \supset g^N \setminus \{ij\} = \left\{g^{N\setminus\{j\}} \cap g^{N\setminus\{i\}}\right\}$. Thus, player i and player j are both veto players for (N, r). □

Proof of Theorem 4.1. For each link $l \in g^N$, let $\mathbf{e^l} \in \mathbb{R}^{n(n-1)/2}$ denote the vector with the l-th coordinate equal to 1 and all other coordinates are 0. By proposition 4.2, to show that the Core of a simple network game is nonempty if and only if there at least two veto players, we only have to show that there is at least one essential link. The proof consists in proving the following equivalence, for (N, r) a simple network game :

$$C(r) = \mathcal{H}\left\{\mathbf{e}^l \in \mathbb{R}^{n(n-1)/2} : l \text{ is an essential link for } r\right\}$$

with $\mathcal{H}$ meaning the convex hull.

1. Suppose that l is an essential link for r. Let $g \subseteq g^N$. If $l \in g$ then $\sum_{l \in g} e^l = 1 \geq r(g)$, otherwise $\sum_{l \in g} e^l = 0 = r(g)$. We know that $r(g^N) = 1 = \sum_{l \in g^N} e^l$. So $\mathbf{e}^l$ is in $C(r)$. This prove the inclusion $\supseteq$ because $C(r)$ is a convex set.

2. Now, let $\mathbf{x} \in C(r)$. It is sufficient to prove that any non-essential link must be allocated 0 according to $\mathbf{x}$. Suppose on the contrary that $l \in g$ is a non-essential link and that $x_l > 0$. Consider g with $r(g) = 1$ and $l \notin g$. Note that such winning network g must exist otherwise l would be essential. Then

$$\sum_{k \in g} x_k = \sum_{k \in g^N} x_k - \sum_{k \in \{g^N \setminus g\}} x_k \leq 1 - x_l < 1$$

contradicting that **x** is a Core element.

□

The last theorem shows that the Core of any simple network game is nonempty only if there are several veto players. Moreover, any Core element allocates the total productivity value of the complete network ($r(g^N) = 1$) only among essential links. Hence, any link-based allocation rule picks up a Core element for simple network game if and only if the total value is distributed only among the veto players.

5 The Shapley value for network games

Discussions in the first sections of this paper showed us how any value generated by cooperation between players can depend on bilateral negotiation. Such situations are adequately modeled by networks, where the set of links between players determines the total productive value of the players collaboration. With the Core of network games, we have seen how to redistribute the productive value of the complete network such that no subseteq of links would be better off, in term of achieving an higher value, in an independant network. But still the Core of a network can be empty or if not, does not single out a solution. In most cases where the Core is nonempty, the Core is set-valued. Another stability concept widely used in the context of network games is to ask the value function to be *component additive*. If the value function is component additive, we are sure that each component of a network receives its due value, i.e. the value the component generates, such that the members of the component don't want to walk away and reallocate their value among themselves. By definition, any allocation rule is component additive, i.e. if the value function is component balanced, then the sum of the payoff allocated to members of a component is the value of the component. In a sense, component balancedness can be seen as a restriction of the Core property. Within the Core, we care about any coalitional deviation, whereas with component balancedness, we only care about possible deviation by components. But, as Jackson [6] puts it (p.138), component balancedness is not sufficient to ensure that an allocation lies in the Core of a network game, but is strong enough to be in conflict with some fairness and anonymity properties.

The aim of this section is to propose a payoff allocation based on some desirable properties. The Shapley value is a one-point solution concept for TU games which is the only one to satisfy a number of properties, or axioms. The Shapley value is well documented in the context of TU games, it has been rediscovered and/or re-characterized many times and has been used in many applications. For these reasons, we found interesting to see whether such solution concept could be transposed to network games. This transposition should arise without too many effort thanks to the associated link-game interpretation we have developed for any network game.

Definition 5.1. *For any network game (N, r), with $r \in \mathcal{R}^N$, a link-based solution on $\mathcal{R}^N$ is a function $\psi : \mathcal{R}^N \to \mathbb{R}^N$ such that there exists an associated function $\psi^l : \mathcal{R}^N \to \mathbb{R}^{n(n-1)/2}$ and $\psi_i(r) = \sum_{j \neq i} \frac{\psi^l_{ij}(r)}{2}$.*

The following properties will help us to characterize the Shapley value as link-based solution in the context of network games. They are the natural transpositions of the original axioms proposed by Shapley [12] for TU games. In the sequel, we consider the player set N as fixed and the network game (or game) is designated by its value function.

Efficiency

For any network game (N, r), with $r \in \mathcal{R}^N$, ψ is an *efficient* link based solution if and only if

$$\sum_{ij \in g^N} \psi^l_{ij}(r) = r(g^N)$$

or equivalently

$$\sum_{i \in N} \psi_i = r(g^N).$$

This property means that a solution for a network game is a feasible vector of real numbers, that is if the sum of the components of the vector is $r(g^N)$. Note that this property is different than the balancedness condition imposed in the definition of an allocation rule. Here, whatever the network the players end up with, the value distributed is the one of the complete network, whereas an allocation rule only distributes the value of the given network that actually forms. For this reason, the characterization of the Shapley value as a (link-based) solution for network games we offer here is different than the characterization of the Shapley value as allocation rule (see Jackson and Wolinsky [8]).

We say that a link ij is *superfluous* in a game (N, r) if $r(g+ij) = r(g)$ for every network $g \subseteq g^N$. This means that the presence of a superfluous link does not affect or contribute anything to any network. Hence, it seems natural that any solution should attribute a zero payoff to such links :

Superfluous link property

For any network game (N, r), with $r \in \mathcal{R}^N$, $\psi^l_{ij}(r) = 0$ for all superfluous links $ij \in g^n$.

We say that two links l and k are symmetric in the game (N, r) if $r(g + l) = r(g + k)$ for any network $g \subseteq g^N \setminus \{l, k\}$. Symmetric links contribute the same amount to any network. Therefore, it seems natural that a solution gives symmetric links the same payoff :

Symmetry

For any network game (N, r), with $r \in \mathcal{R}^N$, $\psi^l_l(r) = \psi^l_k(r)$ for all symmetric links $l, k \in g^n$ in r.

The last property needed in the characterization may be seen as a consistency or decomposability condition. The way the value of a network is allocated may be decomposed so that one may separately allocate the value on different parts of the value function and then sum up :

Additivity

For any network game (N, r) and (N, r'), with $r \in \mathcal{R}^N$ and $r' \in \mathcal{R}^N$, $\psi^l_l(r+r') = \psi^l_l(r)+\psi^l_l(r')$ for any link $l \in g^N$ with $(r + r')(g) \equiv r(g) + r'(g)$.

Definition 5.2. *The Shapley value of a network game* (N, r) *is a* **link-based** *solution that satisfies*

$$\phi_{ij}(r) = \sum_{g:ij\notin g} \frac{l(g)!\,([n(n-1)/2] - l(g) - 1)!}{[n(n-1)/2]!} \left(r(g+ij) - r(g)\right)$$

and

$$\phi_i(r) = \sum_{j\neq i} \frac{\phi_{ij}(r)}{2}.$$

Theorem 5.1. *Let* $\psi : \mathcal{R}^N \to \mathbb{R}^{n(n-1)/2}$ *be a* **link-based** *solution for network games. Then* ψ *satisfies efficiency, superfluous property, symmetry and additivity if and only if* ψ *is the Shapley value* ϕ*.*

Proof. We can see that the Shapley value satisfies the four properties. We have to show that ϕ is the only link-based solution to network games that satisfies the four properties.

Let ψ be a link-based solution that satisfies the four properties. We want to show that $\psi = \phi$. Let $r \in \mathcal{R}^N$. By proposition 2.1, there exist a unique set of numbers c_g ($g \neq \emptyset$) such that $r = \sum_{g \neq \emptyset} c_g r_g$. By additivity of solutions, it follows that

$$\phi(r) = \sum_{g \neq \emptyset} \phi(c_g r_g)$$

and

$$\psi(r) = \sum_{g \neq \emptyset} \psi(c_g r_g).$$

Hence, it is sufficient to show that for any $g \subseteq g^N$, $g \neq \emptyset$ and a scalar c :

$$\psi(cr_g) = \phi(cr_g). \tag{5.1}$$

Chose $g \neq \emptyset$ and a scalar c. Then, for any link $ij \notin g$:

$$cr_g(g' + ij) - cr_g(g') = 0 \qquad \text{for all } g',$$

so that ij is superfluous in cr_g. By the superfluous property of ψ and ϕ :

$$\psi_{ij}(cr_g) = \phi_{ij}(cr_g) = 0 \qquad \text{for all } ij \notin g. \tag{5.2}$$

Now suppose that l and k are links in the network g. Then, for any network $g' \subseteq g^N \setminus \{l, k\}$,

$$cr_g(g' + l) = cr_g(g' + k) = 0$$

implying that links l and k are symmetric in the network game cr_g. Hence by the symmetry property of the link-based solutions ψ and ϕ :

$$\phi_l(cr_g) = \phi_k(cr_g) \qquad \text{for all links } l \text{ and } k \text{ in } g \tag{5.3}$$

and

$$\psi_l(cr_g) = \psi_k(cr_g) \qquad \text{for all links } l \text{ and } k \text{ in } g. \tag{5.4}$$

by efficiency, (5.2), (5.3) and (5.4) we obtain

$$\phi_l(cr_g) = \psi_l(cr_g) = \frac{c}{l(g)} \qquad \text{for all links } l \in g. \tag{5.5}$$

and now, (5.2) and (5.5) imply (5.1). □

6 Conclusions

This paper is a first step into the translation of network games into associated link (TU) games. We adapt the most basic notions of cooperatives games such as the Core and the Shapley value in the context of network games. Some new results about the Core of simple network games are presented. If our approach reveals to be successful, other insights for network games can be reached by the transposition of other well-known notions from cooperative games such as the nucleolus, kernel, convexity issues etc. How the concepts of cooperative games translated into network games behave and are related to concepts of communication games, such as the position value or Myerson value should also be investigated by the mean of more examples. Finally, we also want to develop some solution concepts that exploit the structure of link games per se, without adapting corresponding notions of cooperative games.

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