Collusion in Auctions: between Bid Rotation and Efficiency.*

Wouter Vergote†

Columbia University

October 2004

Abstract

We analyze collusion in an infinitely repeated version of a standard auction with a continuum of types. Because of the lack of efficiency results in this setting the literature has focused on determining and comparing benchmarks on how well bidders can collude. Recent results have shown that the bidders can improve upon static bid rotation. We confirm this result by means of a very simple dynamic mechanism. The novelty of the mechanism lies in the fact that it allows us to determine the maximum amount of collusion it can generate: it recovers one third of the gap between static bid rotation and efficiency.

Keywords: Auctions, Collusion, Repeated Games, Private Information

JEL Classifications: C72, D82

*I'm deeply indebted to Kyle Bagwell and Prajit Dutta for guidance and encouragement. This work also benefited from comments by Atila Abdulkadiroglu, Emanuele Gerratana, Alberto Martin, Filippo Taddei and participants at the Columbia Microeconomics Colloquium.

†Department of Economics, Columbia University, 1022 International Affairs Building, 420 West 118th street, New York, NY 10027. Email: wrv13@columbia.edu

1 Introduction

The possibility that bidders collude in auctions by lowering their bids in a coordinated fashion at the expense of the seller has received ample amount of empirical attention (e.g. Hendricks and Porter (1989), Baldwin et. al. (1997), Pesendorfer (2000), Cramton and Schwartz (2002)). From a theoretical point of view, the main question addressed is whether the best collusive agreement is achievable in equilibrium. This optimal way in which bidders can collude is to allocate the good to the bidder who values it the most without leaving any rent to the seller. The latter is accomplished if the bidder who has the highest valuation bids the reservation price while the other bidders do not participate in the auction. Since the valuation of each bidder is private information it is not hard to understand that such a collusive agreement has little chance of success if there are no enforceable side-payments available or if there are no future auctions in which the bidders participate. In the absence of the latter two features bidders have an incentive to lie about their valuation.

How well bidders manage to collude thus depends on how low they can keep the cost of dealing with the incentive compatibility problem while still allocating the good in an efficient manner. In a static setting this can be achieved at zero cost if binding side-payments are available. However, as these side-payments leave a 'paper trail', which make them easily visible to antitrust authorities, the literature has emphasized that collusion arises in an ongoing relationship (e.g. McAfee and McMillan (1992), Athey and Bagwell (2001), Johnson and Robert (1998), Aoyagi (2003a and b), Skrzypacz and Hopenhayn (2004)). The quest in this literature is to determine whether there exist first best collusive schemes: can bidders, in a repeated environment, obtain an efficient allocation of the good in every period leaving zero rent to the seller if without monetary binding side-payments are available. If not, what does the best collusive scheme look like?

Observe that in a repeated auction without monetary side-payments incentive compatibility is also obtained by transfers. The difference is that the latter are achieved by trading future utility: in order to induce truthtelling today's 'winner' should expect lower continuation utility from tomorrow onwards. An 'efficient' trade of future utility between today's 'winner' and the other bidders can only be realized in the event that this winner and the other bidders (or a subset of them), draw together the highest valuation of all the bidders in some future period. A collusive scheme can then stipulate that the current winner refrains from bidding and allocates the good to one of the other bidders who also have the highest valuation. This way the good is still allocated in an efficient way while lowering the continuation payoff of the current winner. This feature can lead a bidder from claiming too high in the current (or any) period.

It is important to keep in mind that such transfers are restricted to lie in the set of equilibrium continuation values of the repeated auction. Athey and Bagwell (2001), in a repeated Bertrand model that can easily be translated to an auction set-up, show that when the typespace is binary (finite) and satisfies a distributional assumption, the first best can be achieved in equilibrium by trading favors between the bidders. More generally, when the typespace is finite only asymptotic efficiency can be guaranteed. The latter is an immediate consequence of the folk theorem of Fudenberg et al. (1994) for types which are iid distributed. In a repeated auction setting, Aoyagi (2003b) confirmed this result and extended it to affiliated types.

It is not hard too see that when the typespace is a continuum, a very common assumption in the auctions literature, the event that two bidders draw the same type has measure zero and hence incentive compatibility today necessarily implies a loss of efficiency from tomorrow onwards. In this case there is a clear trade-off between efficiency and incentive compatibility. Given that we cannot obtain fully efficient collusion a natural question to ask is how well bidders can collude in an equilibrium of the repeated auctions game.

In an influential paper, McAfee and McMillan (1992) show that, when binding side payments are not available, a bidding cartel or bidding ring can outperform the static Bayesian Nash equilibrium. A bidding ring occurs when all (or a subset) of the bidders engage in bid rigging in order to lower the price. One example is bid rotation: all players submit bids but they rotate who is allowed to have the highest bid and obtain the good[1]. In the static bid rotation scheme with two bidder for example, a bidder obtains the good every other period or obtains the good each period with 50% chance. In such a mechanism there is no truthtelling problem: the winner of the auction is determined regardless of her claim (valuation). The drawback of this mechanism is that, in expectation, the good is allocated in an efficient way only half of the times. McAfee and McMillan also stress that there is no incentive to adhere to such a collusive agreement if there is no punishment

[1] Other examples are submitting identical bids, bid suppression, unacceptable bids and clearly combinations of these. All of these examples limit the competition between the bidders. When bids are rigged there is no incentive problem as winning is independent of valuation.

for violating it openly. They subsequently assume an unmodeled repeated relationship in which deviation is deterred through future equilibrium punishment.

At the beginning of the nineties a rich literature on repeated games with private information developed. In two seminal papers, Abrue et al. (1986, 1990) greatly simplified the characterization of (perfect public) equilibria (PPE) of these games by relying on dynamic programming techniques. Following this approach allows each bidder's PPE payoff to be factored into two components: current-period payoffs and (discounted) continuation values. In this sense, PPE continuation values can be interpreted as playing a role similar to that of side payments, although this set of transfers is restricted by equilibrium considerations. Aoyagi (2003a) builds upon this literature and demonstrates the existence of a dynamic mechanism that outperforms the static bid rotation scheme proposed by McAfee and McMillan (1992). Nonetheless, his mechanism does not allow, in general, to pin down exactly how much better bidders can do in equilibrium.

The purpose of the present paper is to propose, in a similar set-up, a very simple mechanism that also improves upon static bid rotation but at the same time allows us to exactly characterize how much more efficiently this mechanism allows the bidders to collude. We first do so for the 2-bidder environment presented in Aoyagi (2003)[2]. In this setting we develop a collusive mechanism by introducing two states: a punishment state and a reward state. If one bidder is in the punishment state the other is in the reward state and vice versa. In each state the bidders announce their claims and the mechanism instructs, with probability φ, the bidder with the highest claim to bid the reservation price and the other bidder to stay out. With probability $(1-\varphi)$ the punished player is told to stay out while the other obtains the good at the reservation price. Additionally, the claims of the bidders determine, probabilistically, the state they will be in tomorrow. A higher claim leads to a higher probability of being punished tomorrow. The latter feature of the mechanism guarantees incentive compatiblity. We determine an upper bound on φ for which there exist a patience level such that the above collusive scheme can be supported as an equilibrium of the repeated auctions game with communication. The virtue of our mechanism is twofold. First, it is very simple as we only need two states and adjustments in continuation values are achieved in one period. Second,

[2] We do so for the case without affiliated types as this would unnecessarily complicate the main message we wish to convey.

the mechanism allows us to determine an upper bound on how much collusion it can sustain in equilibrium.

The rest of the paper is organized as follows : Section 2 discusses the static setup and provides basic notation. Section 3, the repeated auction is introduced together with the collusive mechanism we propose. Section 4 contains the main result of the paper: it defines the conditions for our mechanism to be supported as a PPE. Section 5 concludes. An appendix provides the proofs.

2 Stage Game Auction

We assume that there are two bidders. We denote bidder one with (i) and bidder two with (j). We focus on the independent private value case (IPV) which assumes the bidders are ex-ante symmetric and draw an independent private value for the good from a common continuous distribution F with strictly positive continuously differentiable density f and support $\Theta = [0,1]$. We assume that F satisfies the following hazard rate condition: $h'(\theta) < 0$ where $h(\theta) = \frac{1-F(\theta)}{f(\theta)}$. We allow for the fact that one or all bidders do not participate in the auction. Hence the bidders choose a bid from the set $B = \{\varnothing \cup R^{+}\}$. We assume for simplicity that the seller's reservation price equals zero. In what will follow we will focus on a first price sealed bid auction but it will be straightforward to see that our reasoning holds for any auctioning rule used by the auctioneer such that:

- The highest bidder obtains good. The other bidders does not pay a transfer to the seller. When there is a tie, the good is allocated randomly with equal probability to any of the two bidders. Again, this is done at the bidding price for the winner and the loser pays zero.

- If nobody bids, the good remains in the hands of the seller.

The expected payoff of efficient collusion could be written as v^* where :

$$v^* = \int_0^1 \theta F(\theta) f(\theta) d\theta$$

The expected payoff of a bid rotation (McAfee and McMillan (1992)), is the same as when each bidder obtains the good with equal probability. We label it $\frac{\bar{v}}{2}$ where

$$\bar{v} = E(\theta) = \int_0^1 \theta f(\theta) d\theta$$

It can be shown that there exists a symmetric Bayesian-Nash equilibrium for this game with expected payoffs equal to v^N. Given the assumption on h we have that $v^N < \frac{\bar{v}}{2}$. In the appendix we show that $v^* > \frac{\bar{v}}{2}$ and hence we have that

$$v^* > \frac{\bar{v}}{2} > v^N$$

3 The Repeated Auction

3.1 Setup

In the repeated game we assume that the bidders' private values are iid over time and we allow for pre-play communication in each period. Communication is introduced by assuming that the players have access to a communication device: the *center.* The task of the latter is to collect the bidders' claims, and on the basis of these to recommend each bidder how much to bid[3].

Coordination through communication is then modelled as follows. In each period t the bidders play the following stage game:

1. Each bidder $i = 1, 2$ observes her types θ_i^t. For our exposition we do not need the time superscripts and hence omit them below.

2. Each bidder $i = 1, 2$ makes an announcement to the mechanism denote by $\hat{\theta}_i(\theta_i)$, where $\hat{\theta}_i$ is

[3] We introduce the idea of a communication center for ease of exposition and to work with a comparable set-up to that of Aoyagi (2003a) [3]. We could do without the center by letting the players, when announcing their types, also propose: a) a bidding rule based on the announcements and the outcome of the randomization device, and; b) an adjustment rule governing the probabilities used for randomization as a function of announcements. The bidders would then effectively assume the role of the communication device. Such a set-up would be similar to that of Athey and Bagwell [4].

the announcement rule

$$\hat{\theta}_i(.) : \Theta \to \Theta$$

3. In the collusive stage of the mechanism there two possible states: P and R. In state P bidder i is punished and bidder j is rewarded. In state R exactly the opposite holds. Hence the state space is just $S = \{P, R\}$ where the state refers to the state of bidder i. Given the current state and announcements the mechanism instructs each bidder how much to bid using the instruction rule $m : S \times [0,1]^2 \to B^2$ where m is defined by

$$\begin{aligned} \text{in state } P \quad &: \quad \text{with probability } \varphi \text{ the bidder with the highest claim obtains the good} \\ &\quad\ \text{with probability } 1-\varphi \text{ bidder } j \text{ obtains the good regardless of her claim} \end{aligned}$$

$$\begin{aligned} \text{in state } R \quad &: \quad \text{with probability } \varphi \text{ the bidder with the highest claim obtains the good} \\ &\quad\ \text{with probability } 1-\varphi \text{ bidder } i \text{ obtains the good regardless of her claim} \end{aligned}$$

4. Given the claims of both bidders there is a transition rule to tomorrow's state, independent of today's state. This is accomplished by the fact that the claims determine the probability $\pi : \Theta^2 \to [0,1]$ that tomorrow's state is P (bidder i punished and bidder j rewarded). We call π the transition rule.

5. We define M to be a collection of the assignment rule m and transition rule π. In short

$$M = \{m, \pi\}$$

After observing the recommendation of the mechanism and depending on what her true valuation for the good is, each bidder places her bid according to a bidding rule $\hat{b}_i$ $i = 1, 2$ where

$$\hat{b}_i(.) : B \times \Theta \to B$$

Moreover, let θ_i where $i = 1, 2$ be the honest reporting rule for bidder one (bidder two)

$$\theta_i(x) = x \qquad \text{for all } x \in \Theta,\ i = 1, 2$$

and let b_i, $i = 1, 2$, be the obedient bidding rules (bidders follow the mechanism's instructions) so that

$$b_i(m(\hat{\theta}_i, \hat{\theta}_j, s), \theta_i) = m_i(\hat{\theta}_i, \hat{\theta}_j, s) \text{ where } j \neq i$$

We assume that the bidders decide on the rules of the mechanism at time zero. The mechanism is assumed to begin in a "collusive phase": at time zero the state is chosen at random after which it is determined by the claims of the bidders. After any observable deviation the mechanism reverts "non-collusive phase" which is characterized by playing the Bayesian Nash equilibrium forever. In the latter bidders obtain v^N per period in expectation.

3.2 The Mechanism as a Perfect Public Equilibrium

Let $\pi_i(\hat{\theta}, \hat{b}, M)$ denote bidder $i's$ expected payoff (bidder $j's$ is defined analogously) from the stage game as a function of the announcement, bidding and instruction rules. Communication history for a bidder in period t in the repeated game is the sequence of his announcements and instructions in periods 1, 2, ..., $t-1$. Private history is the sequence of its private signals θ_{ik} in periods $k = 1, 2, ..., t-1$. Finally, public history in period t is a sequence of outcomes of the assignment rule used by the mechanism, the actual bids and communication history.

Bidder $i's$ strategy $\hat{\sigma}_i$ is a pair of announcement and bidding rules $(\hat{\theta}_i, \hat{b}_i)$ for each period defined as a function of his public and private histories. Define σ to be the honest and obedient strategy which selects the pair (θ, b) for all histories. Bidders aim to maximize their expected discounted payoff given a common discount factor $\delta < 1$. $M = \{m(\varphi), \pi\}$ is a sequential equilibrium of the repeated auction if the pair $\Sigma = (\sigma_i, \sigma_j)$ is a perfect public equilibrium (PPE) of the repeated game, i.e., if σ_i is optimal against (σ_j, M) after any public history of the game. That is, what is required is that bidders are truthful and obedient.

4 The Main Result

We will now show our main result: one can find a probability mapping π such that $M = \{m, \pi\}$ can be supported as an equilibrium of the repeated auctions game for a high enough level of patience. To do so, let us define expected welfare in state P to be W^P for bidder i (the punished) and W^R for bidder j (the rewarded). Assume for now - we will below show the conditions under which this holds - that there exists an incentive compatible transition mapping $\pi(\theta) = \pi(\hat{\theta}_i, \hat{\theta}_j)$. Then expected payoffs in each state can be written recursively as:

$$\begin{aligned} \text{for bidder } i \quad W^R &= (1-\delta)(\varphi v^* + (1-\varphi)E\theta) + \delta(\pi W^P + (1-\pi)W^R) && (1) \\ W^P &= (1-\delta)\varphi v^* + \delta(\pi W^P + (1-\pi)W^R) && (2) \end{aligned}$$

$$\begin{aligned} \text{for bidder } j \quad W^R &= (1-\delta)(\varphi v^* + (1-\varphi)E\theta) + \delta(\pi W^R + (1-\pi)W^P) && (3) \\ W^P &= (1-\delta)\varphi v^* + \delta(\pi W^R + (1-\pi)W^P) && (4) \end{aligned}$$

where $\pi = \int_0^1 \int_0^1 \pi(\theta) f(\theta_i) f(\theta_j) d\theta_i d\theta_j$. From the above we have that:

$$W^R - W^P = (1-\delta)(1-\varphi)E\theta = (1-\delta)(1-\varphi)\bar{v}$$

Interim welfare for bidder one after observing her valuation and given a truthful and obedient strategy of bidder two is:

$$\begin{aligned} W^R(\theta_i, \hat{\theta}_i) &= (1-\delta)(\varphi\theta_i F(\hat{\theta}_i) + (1-\varphi)\theta_i) + \delta \int_0^1 [\pi(\hat{\theta}_i, \theta_j) W^P + (1-\pi(\hat{\theta}_i, \theta_j)) W^R] f(\theta_j) d\theta_j && (5) \\ W^B(\theta_i, \hat{\theta}_i) &= (1-\delta)\varphi\theta_i F(\hat{\theta}_i) + \delta \int_0^1 [\pi(\hat{\theta}_i, \theta_j) W^P + (1-\pi(\hat{\theta}_i, \theta_j)) W^R] f(\theta_j) d\theta_j && (6) \end{aligned}$$

For bidder two (j) this means

$$W^R(\theta_j, \hat{\theta}_j) = (1-\delta)(\varphi\theta_j F(\hat{\theta}_j) + (1-\varphi)\theta_j) + \delta \int_0^1 [\pi(\theta_i, \hat{\theta}_j) W^R + (1-\pi(\theta_i, \hat{\theta}_j)) W^P] f(\theta_i) d\theta_i \quad (7)$$

$$W^R(\theta_j, \hat{\theta}_j) = (1-\delta)\varphi\theta_j F(\hat{\theta}_j) + \delta \int_0^1 [\pi(\theta_i, \hat{\theta}_j) W^R + (1-\pi(\theta_i, \hat{\theta}_j)) W^P] f(\theta_i) d\theta_i \quad (8)$$

The transition rule π induced local incentive compayibility if:

$$\frac{W^R(\theta_i, \hat{\theta}_i)}{\partial \hat{\theta}_i} \mid_{\hat{\theta}_i = \theta_i} = 0 \text{ and } \frac{W^P(\theta_i, \hat{\theta}_i)}{\partial \hat{\theta}_i} \mid_{\hat{\theta}_i = \theta_i} = 0$$

The same holds true for bidder j.We then have the following lemma:

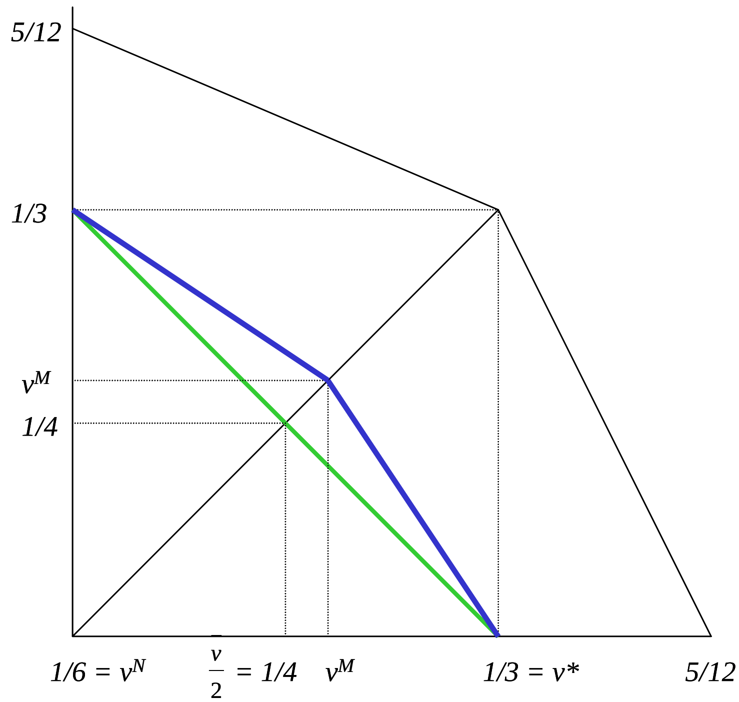
Lemma 1 *Let* $\varphi < \frac{1}{3}$, *and let* $\pi(\theta_i, \theta_j) = \frac{1}{2} - \frac{(1-\delta)\varphi}{\delta(W^R - W^P)} \int_0^{\theta_i} \theta f(\theta) d\theta + \frac{(1-\delta)\varphi}{\delta(W^R - W^P)} \int_0^{\theta_j} \theta f(\theta) d\theta$. *Then there exists a level of patience* $\delta(\varphi) = \frac{2\varphi}{(1-\varphi)} < 1$ *such that for all* $\delta \geq \delta(\varphi)$ *our mechanism* $M = \{m(\varphi), \pi\}$ *is incentive compatible (locally as well as globally).*

Proof. See Appendix ■

The above defined probability mapping guarantees that the bidders will always announce their valuation in a truthful manner locally. The single crossing property then guarantees us that incentive compatibility is also satisfied globally. Note that, as can be expected, the probability of moving to their punishment state is increasing in the bidders' announcements. A higher announcement today is thus punished with expected lower payoffs in the next period. This means that the whole transfer needed to obtain incentive compatibility is obtained in the next period only. The cost, however, is that the good is allocated in an efficient way only with probability φ. We would like to stress that this result is general in the sense tha φ is independent from the support of the distribution F.

What remains to show however, is that no bidder ever has an incentive to openly deviate from the instructions given by the mechanism. The following proposition tells us when such off-schedule deviations are deterred.

Proposition 2 *Let* $M = \{m(\varphi), \pi\}$ *where* $\varphi < \frac{1}{3}$, *and* π *is defined as above. Then there exist a*



$\delta^{NR}(M)$ such that M is immune for off-schedule deviations. Then let $\delta^* = \max\{\delta^{NR}(M), \delta(\varphi)\}$. For all $\delta > \delta^*$ $M = \{m(\varphi), \pi\}$ is an equilibrium of the repeated auctions game. The expected payoff of this equilibrium is $\varphi v^* + (1-\varphi)\frac{\bar{v}}{2}$. For $\varphi \to \frac{1}{3}$ we need $\delta \to 1$ and the expected payoff converges to $\frac{1}{3}v^* + \frac{2}{3}\frac{\bar{v}}{2}$.

Proof. See Appendix ■

This proposition confirms that our mechanism can, with patient enough bidders, recover one third of the gap between bid rotation and efficiency.

5 A finite amount of bidders

We now investigate how we can extend the above mechanism in the case that there are any finite number (N) of bidders. We will show that the intuition developed for the two bidder case naturally carries over to a situation with more bidders. We do so by developing a mechanism in which the bidder with the highest valuation faces the possibility if moving into a punishment phase, while all the other bidders enter a punishing phase. In that phase with a given probability, φ, the punished bidder is told not to bid while the other bidders play distribute the good among them using bid rotation.

when deciding what to announce in the punishment phase:

$$\text{for bidder } i \quad W^R = (1-\delta)(\varphi v^* + (1-\varphi)\frac{E\theta}{n-1}) + \delta(\pi W^P + (1-\pi)W^R) \tag{9}$$

$$W^P = (1-\delta)\varphi v^* + \delta(\pi W^P + (1-\pi)W^R) \tag{10}$$

$$\text{for bidder } j \quad W^R = (1-\delta)(\varphi v^* + (1-\varphi)E\theta) + \delta(\pi W^R + (1-\pi)W^P) \tag{11}$$

$$W^P = (1-\delta)\varphi v^* + \delta(\pi W^R + (1-\pi)W^P) \tag{12}$$

where $\pi = \int_0^1 ... \int_0^1 \pi(\theta) f(\theta_1)...f(\theta_n) d\theta_1 ... d\theta_n$. From the above we have that:

$$W^R - W^P = (1-\delta)(1-\varphi)\frac{E\theta}{n-1} = (1-\delta)(1-\varphi)\frac{\bar{v}}{n-1}$$

Interim welfare for bidder one after observing her valuation and given a truthful and obedient strategy of bidder i is:

$$W^R(\theta_i, \hat{\theta}_i) = (1-\delta)(\varphi\theta_i F(\hat{\theta}_i)^{n-1} + (1-\varphi)\frac{\theta_i}{n-1}) + \delta \int_0^1 ... \int_0^1 [\pi(\hat{\theta}_i, \theta_{-i})W^P + (1-\pi(\hat{\theta}_i, \theta_{-i}))W^R] f(\theta_1^{-i})...f(\theta$$

$$W^P(\theta_i, \hat{\theta}_i) = (1-\delta)\varphi\theta_i F(\hat{\theta}_i)^{n-1} + \delta \int_0^1 ... \int_0^1 [\pi(\hat{\theta}_i, \theta_{-i})W^P + (1-\pi(\hat{\theta}_i, \theta_{-i}))W^R] f(\theta_1^{-i})...f(\theta_n^{-i}) d\theta_1^{-i} ... d\theta_n^{-i}$$

For bidder two (j) this means

$$W^R(\theta_j,\hat{\theta}_j) = (1-\delta)(\varphi\theta_j F(\hat{\theta}_j)+(1-\varphi)\theta_j)+\delta\int_0^1[\pi(\theta_i,\hat{\theta}_j)W^R+(1-\pi(\theta_i,\hat{\theta}_j))W^P]f(\theta_i)d\theta_i \quad (15)$$

$$W^P(\theta_j,\hat{\theta}_j) = (1-\delta)\varphi\theta_j F(\hat{\theta}_j)+\delta\int_0^1[\pi(\theta_i,\hat{\theta}_j)W^R+(1-\pi(\theta_i,\hat{\theta}_j))W^P]f(\theta_i)d\theta_i \quad (16)$$

The transition rule π induced local incentive compayibility if:

$$\frac{W^R(\theta_i,\hat{\theta}_i)}{\partial\hat{\theta}_i}\,|_{\hat{\theta}_i=\theta_i}=0 \text{ and } \frac{W^P(\theta_i,\hat{\theta}_i)}{\partial\hat{\theta}_i}\,|_{\hat{\theta}_i=\theta_i}=0$$

The same holds true for bidder j.We then have the following lemma:

$\frac{W^R(\theta_i,\hat{\theta}_i)}{\partial\hat{\theta}_i}\,|_{\hat{\theta}_i=\theta_i}=0$ implies:

$$(1-\delta)\varphi\theta_i(n-1)F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i)=\delta(W^R-W^P)\int_0^1...\int_0^1\frac{\partial\pi_i(\hat{\theta}_i,\theta_{-i})}{\partial\hat{\theta}_i}f(\theta_1^{-i})...f(\theta_n^{-i})d\theta_1^{-i}...d\theta_n^{-i}$$

$$\begin{aligned}\int_0^1...\int_0^1\frac{\partial\pi_i(\hat{\theta}_i,\theta_{-i})}{\partial\hat{\theta}_i}f(\theta_1^{-i})...f(\theta_n^{-i})d\theta_1^{-i}...d\theta_n^{-i} &= \frac{(1-\delta)\varphi\theta_i(n-1)F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i)}{\delta(W^R-W^P)}\\ &= \frac{(1-\delta)\varphi\theta_i(n-1)F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i)}{\delta(1-\delta)(1-\varphi)\frac{\bar{v}}{n-1}}\\ &= \frac{\varphi\theta_i(n-1)^2F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i)}{\delta(1-\varphi)\bar{v}}\end{aligned}$$

Now let us try an additive form for π :

$$\pi_i(\theta_1, ..., \theta_n) = c + \frac{\varphi}{\delta(1-\varphi)\bar{v}} \int_0^{\theta_i} \theta_i (n-1)^2 F(\theta_i)^{n-2} f(\theta_i) d\theta_i - \frac{1}{n-1} \sum_{j \neq i}^{n} \frac{\varphi}{\delta(1-\varphi)\bar{v}} \int_0^{\theta_j} \theta_j (n-1)^2 F(\theta_j)^{n-2} f($$

$$\pi_i(\theta_1, ..., \theta_n) = c + \sum_{j=1}^{n} \pi_i^j \text{ where}$$

$$\pi_i^i = \frac{\varphi}{\delta(1-\varphi)\bar{v}} \int_0^{\theta_i} \theta_i (n-1)^2 F(\theta_i)^{n-2} f(\theta_i) d\theta_i$$

$$\pi_i^j = -\frac{1}{n-1} \frac{\varphi}{\delta(1-\varphi)\bar{v}} \int_0^{\theta_j} \theta_j (n-1)^2 F(\theta_j)^{n-2} f(\theta_j) d\theta_j$$

Then it is the case that:

$$\frac{\partial \pi_i}{\partial \theta_i} = \frac{\varphi}{\delta(1-\varphi)\bar{v}} \frac{\partial \int_0^{\theta_i} \theta_i (n-1)^2 F(\theta_i)^{n-2} f(\theta_i) d\theta_i}{\partial \theta_i}$$

$$\frac{\partial \pi_j}{\partial \theta_i} = -\frac{1}{n-1} \frac{\varphi}{\delta(1-\varphi)\bar{v}} \frac{\partial \int_0^{\theta_i} \theta_i (n-1)^2 F(\theta_i)^{n-2} f(\theta_i) d\theta_i}{\partial \theta_i}$$

and

$$\int_0^1 ... \int_0^1 \frac{\partial \pi_i}{\partial \theta_i} f(\theta_1^{-i}) ... f(\theta_n^{-i}) d\theta_1^{-i} ... d\theta_n^{-i} = \frac{\partial \pi_i^i}{\partial \theta_i}$$

Hence:

$$\frac{\partial \pi_i^i}{\partial \theta_i} = \frac{\varphi \theta_i (n-1)^2 F(\hat{\theta}_i)^{n-2} f(\hat{\theta}_i)}{\delta(1-\varphi)\bar{v}}$$

$$\pi_i^i(\theta_i) = \frac{\int_0^{\theta_i} \varphi x (n-1)^2 F(x)^{n-2} f(x) dx}{\delta(1-\varphi)\bar{v}}$$

$$= \frac{\varphi(n-1)^2}{\delta(1-\varphi)\bar{v}} \int_0^{\theta_i} x F(x)^{n-2} f(x) dx$$

$$= \frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}} \left[\theta_i F(\theta_i)^{n-1} - \int_0^{\theta_i} F(x)^{n-1} dx \right]$$

in expectations:

$$E\pi_i = c + E\sum_{j=1}^{n} \pi_i^j$$

where

$$\begin{aligned} E\pi_i^i &= \frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}}\left[\int_0^1 \theta_i F(\theta_i)^{n-1} f(\theta_i)d\theta_i - \int_0^1\int_0^{\theta_i} F(x)^{n-1}dx f(\theta_i)d\theta_i\right] \\ E\pi_i^j &= -\frac{1}{n-1}\frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}}\left[\int_0^1 \theta_j F(\theta_j)^{n-1} f(\theta_j)d\theta_j - \int_0^1\int_0^{\theta_j} F(x)^{n-1}dx f(\theta_j)d\theta_j\right] \end{aligned}$$

Hence

$$E\pi_i = c$$

We impose symmetry and hence

$$E\pi_i = c = \frac{1}{n}$$

We now need to impose some restrictions on φ for our mechanism to work. This is the case when $0 \leq \pi_i(\theta) \leq 1$ for all i and for all θ. Moreover we impose that $0 \leq \delta < 1$.

In order to do so we need to check the following two inequalities:

$$\begin{aligned} \pi_i(\theta_i, \theta_{-i}) &\leq 1 \\ \pi_i(\theta_i, \theta_{-i}) &\geq 0 \end{aligned}$$

It can easily be checked that the second inequality is stonger than the first one (see appendix) and hence we only need to check that: $\pi_i(0, \mathbf{1}) \geq 0$ and this is translated into:

$$
\begin{aligned}
\pi_i(1,\mathbf{0}) &= \frac{1}{n} - \frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}}\left[1-\int_0^1 F(x)^{n-1}dx\right] \geq 0 \\
& \quad \textit{or} \\
\frac{\varphi}{\delta(1-\varphi)\bar{v}}\left[1-\int_0^1 F(x)^{n-1}dx\right] &\leq \frac{1}{n(n-1)} \\
\frac{\varphi}{\delta(1-\varphi)\bar{v}}\left[1-\int_0^1 F(x)^{n-1}dx\right] &\leq \frac{1}{n(n-1)} \\
\frac{\varphi\Delta}{\delta(1-\varphi)\bar{v}} &\leq \frac{1}{n} \text{ where } \Delta = \left[1-\int_0^1 F(x)^{n-1}dx\right] \\
\frac{\varphi}{1-\varphi} &\leq \frac{\delta\bar{v}}{n\Delta} \\
\frac{\varphi}{1-\varphi} &< \frac{\bar{v}}{n\Delta} \text{ since } \delta < 1 \\
\varphi &< (1-\varphi)\frac{\bar{v}}{n\Delta} \\
\varphi\left(1+\frac{\bar{v}}{n\Delta}\right) &< \frac{\bar{v}}{n\Delta} \\
\varphi &< \frac{\bar{v}}{n(n-1)\Delta+\bar{v}} = \psi(n)
\end{aligned}
$$

Note that when $n = 2$ we recover the result of the previous section: it's easy to show that then $\Delta = \bar{v}$ and hence $\psi(n) = \frac{1}{3}$. For $n > 2 : \psi(n) > \psi(n+1)$.

Lemma 3 *Let* $\varphi < \frac{1}{3}$, *and let* $\pi(\theta_i, \theta_j) = \frac{1}{2} - \frac{(1-\delta)\varphi}{\delta(W^R - W^P)}\int_0^{\theta_i}\theta f(\theta)d\theta + \frac{(1-\delta)\varphi}{\delta(W^R-W^P)}\int_0^{\theta_j}\theta f(\theta)d\theta$. *Then there exists a level of patience* $\delta(\varphi) = \frac{2\varphi}{(1-\varphi)} < 1$ *such that for all* $\delta \geq \delta(\varphi)$ *our mechanism* $M = \{m(\varphi), \pi\}$ *is incentive compatible (locally as well as globally).*

Proof. See Appendix ■

The above defined probability mapping guarantees that the bidders will always announce their valuation in a truthful manner locally. The single crossing property then guarantees us that incentive compatibility is also satisfied globally. Note that, as can be expected, the probability of moving to their punishment state is increasing in the bidders' announcements. A higher announcement today is thus punished with expected lower payoffs in the next period. This means that the whole transfer needed to obtain incentive compatibility is obtained in the next period only. The cost, however, is that the good is allocated in an efficient way only with probability φ. We would like to stress that this result is general in the sense tha φ is independent from the support of the distribution F.

What remains to show however, is that no bidder ever has an incentive to openly deviate from the instructions given by the mechanism. The following proposition tells us when such off-schedule deviations are deterred.

Proposition 4 *Let $M = \{m(\varphi), \pi\}$ where $\varphi < \frac{1}{3}$, and π is defined as above. Then there exist a $\delta^{NR}(M)$ such that M is immune for off-schedule deviations. Then let $\delta^* = \max\{\delta^{NR}(M), \delta(\varphi)\}$. For all $\delta > \delta^*$ $M = \{m(\varphi), \pi\}$ is an equilibrium of the repeated auctions game. The expected payoff of this equilibrium is $\varphi v^* + (1-\varphi)\frac{\bar{v}}{2}$. For $\varphi \to \frac{1}{3}$ we need $\delta \to 1$ and the expected payoff converges to $\frac{1}{3}v^* + \frac{2}{3}\frac{\bar{v}}{2}$.*

Proof. See Appendix ■

This proposition confirms that our mechanism can, with patient enough bidders, recover one third of the gap between bid rotation and efficiency.

6 Efficiency

Let us now denote the expected payoff for a given bidder i in either the reward or the punishment phase:

$$\text{for bidder } i \quad W_i^R = (1-\delta)(\varphi v_n^* + (1-\varphi)v_{n-1}^*) + \delta(\pi^R W^P + (1-\pi^R)W^R) \tag{17}$$

$$W_i^P = (1-\delta)\varphi v_n^* + \delta(\pi^P W^P + (1-\pi^P)W^R) \tag{18}$$

Where π_i^R is the expected probability of entering the punishment phase next period before the realization of today's valuation when bidder i is currently one of the rewarded bidders. If bidder i is punished then π_i^P is the expected probability of entering the punishment phase next period. In order for this mechanism to be consistent we impose that for every realization of the valuations, $\theta = (\theta_1, ..., \theta_n)$ and for a given punished player j:

$$\pi_j^P(\theta) + \sum_{i \neq j}^{n} \pi_i^R(\theta) = 1$$

We then have that

$$\begin{aligned}
W^R - W^P &= (1-\delta)(1-\varphi)v_{n-1}^* + \delta(\pi^R - \pi^P)(W^P - W^R) \\
W^R - W^P &= \frac{(1-\delta)(1-\varphi)v_{n-1}^*}{1+\delta(\pi^R - \pi^P)} = \frac{(1-\delta)(1-\varphi)v_{n-1}^*}{1+\delta(n\pi^R - 1)}
\end{aligned}$$

where $\pi = \int_0^1 ... \int_0^1 \pi(\theta) f(\theta_1)...f(\theta_n) d\theta_1 ... d\theta_n$. From the above we have that:Interim welfare for bidder one after observing her valuation and given a truthful and obedient strategy of bidder i is:

$$\begin{aligned}
W^R(\theta_i, \hat{\theta}_i) &= (1-\delta)(\varphi\theta_i F(\hat{\theta}_i)^{n-1} + (1-\varphi)\theta_i F(\hat{\theta}_i)^{n-2}) + \delta \int_0^1 ... \int_0^1 [\pi^R(\hat{\theta}_i, \theta_{-i}) W^P + (1 - \pi^R(\hat{\theta}_i, \theta_{-i})) W^R] f(\theta \\
W^P(\theta_i, \hat{\theta}_i) &= (1-\delta)\varphi\theta_i F(\hat{\theta}_i)^{n-1} + \delta \int_0^1 ... \int_0^1 [\pi^P(\hat{\theta}_i, \theta_{-i}) W^P + (1 - \pi^P(\hat{\theta}_i, \theta_{-i})) W^R] f(\theta_1^{-i})...f(\theta_n^{-i}) d\theta_1^{-i} ... d\theta_n^{-i}
\end{aligned}$$

For bidder two (j) this means

$$W^R(\theta_j, \hat{\theta}_j) = (1-\delta)(\varphi\theta_j F(\hat{\theta}_j) + (1-\varphi)\theta_j) + \delta \int_0^1 [\pi(\theta_i, \hat{\theta}_j) W^R + (1 - \pi(\theta_i, \hat{\theta}_j)) W^P] f(\theta_i) d\theta_i \tag{21}$$

$$W^P(\theta_j, \hat{\theta}_j) = (1-\delta)\varphi\theta_j F(\hat{\theta}_j) + \delta \int_0^1 [\pi(\theta_i, \hat{\theta}_j) W^R + (1 - \pi(\theta_i, \hat{\theta}_j)) W^P] f(\theta_i) d\theta_i \tag{22}$$

The transition rule π induced local incentive compayibility if:

$$\frac{W^R(\theta_i, \hat{\theta}_i)}{\partial \hat{\theta}_i} |_{\hat{\theta}_i = \theta_i} = 0 \text{ and } \frac{W^P(\theta_i, \hat{\theta}_i)}{\partial \hat{\theta}_i} |_{\hat{\theta}_i = \theta_i} = 0$$

The same holds true for bidder j.We then have the following lemma:

$\frac{W^R(\theta_i,\hat{\theta}_i)}{\partial\hat{\theta}_i} \mid_{\hat{\theta}_i=\theta_i} = 0$ implies:

$$(1-\delta)\varphi\theta_i(n-1)F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i) = \delta(W^R - W^P)\int_0^1 ... \int_0^1 \frac{\partial\pi_i(\hat{\theta}_i,\theta_{-i})}{\partial\hat{\theta}_i} f(\theta_1^{-i})...f(\theta_n^{-i})d\theta_1^{-i}...d\theta_n^{-i}$$

$$\begin{aligned}
\int_0^1 ... \int_0^1 \frac{\partial\pi_i(\hat{\theta}_i,\theta_{-i})}{\partial\hat{\theta}_i} f(\theta_1^{-i})...f(\theta_n^{-i})d\theta_1^{-i}...d\theta_n^{-i} &= \frac{(1-\delta)\varphi\theta_i(n-1)F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i)}{\delta(W^R-W^P)} \\
&= \frac{(1-\delta)\varphi\theta_i(n-1)F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i)}{\delta(1-\delta)(1-\varphi)\frac{\bar{v}}{n-1}} \\
&= \frac{\varphi\theta_i(n-1)^2F(\hat{\theta}_i)^{n-2}f(\hat{\theta}_i)}{\delta(1-\varphi)\bar{v}}
\end{aligned}$$

Now let us try an additive form for π :

$$\begin{aligned}
\pi_i(\theta_1,....,\theta_n) &= c + \frac{\varphi}{\delta(1-\varphi)\bar{v}}\int_0^{\theta_i}\theta_i(n-1)^2F(\theta_i)^{n-2}f(\theta_i)d\theta_i - \frac{1}{n-1}\sum_{j\neq i}^{n}\frac{\varphi}{\delta(1-\varphi)\bar{v}}\int_0^{\theta_j}\theta_j(n-1)^2F(\theta_j)^{n-2}f(\\
\pi_i(\theta_1,....,\theta_n) &= c + \sum_{j=1}^{n}\pi_i^j \text{ where} \\
\pi_i^i &= \frac{\varphi}{\delta(1-\varphi)\bar{v}}\int_0^{\theta_i}\theta_i(n-1)^2F(\theta_i)^{n-2}f(\theta_i)d\theta_i \\
\pi_i^j &= -\frac{1}{n-1}\frac{\varphi}{\delta(1-\varphi)\bar{v}}\int_0^{\theta_j}\theta_j(n-1)^2F(\theta_j)^{n-2}f(\theta_j)d\theta_j
\end{aligned}$$

Then it is the case that:

$$\begin{aligned}
\frac{\partial\pi_i}{\partial\theta_i} &= \frac{\varphi}{\delta(1-\varphi)\bar{v}}\frac{\partial\int_0^{\theta_i}\theta_i(n-1)^2F(\theta_i)^{n-2}f(\theta_i)d\theta_i}{\partial\theta_i} \\
\frac{\partial\pi_j}{\partial\theta_i} &= -\frac{1}{n-1}\frac{\varphi}{\delta(1-\varphi)\bar{v}}\frac{\partial\int_0^{\theta_i}\theta_i(n-1)^2F(\theta_i)^{n-2}f(\theta_i)d\theta_i}{\partial\theta_i}
\end{aligned}$$

and

$$\int_0^1 ... \int_0^1 \frac{\partial \pi_i}{\partial \theta_i} f(\theta_1^{-i})...f(\theta_n^{-i}) d\theta_1^{-i}...d\theta_n^{-i} = \frac{\partial \pi_i^i}{\partial \theta_i}$$

Hence:

$$\begin{aligned}
\frac{\partial \pi_i^i}{\partial \theta_i} &= \frac{\varphi \theta_i (n-1)^2 F(\hat{\theta}_i)^{n-2} f(\hat{\theta}_i)}{\delta(1-\varphi)\bar{v}} \\
\pi_i^i(\theta_i) &= \frac{\int_0^{\theta_i} \varphi x (n-1)^2 F(x)^{n-2} f(x) dx}{\delta(1-\varphi)\bar{v}} \\
&= \frac{\varphi (n-1)^2}{\delta(1-\varphi)\bar{v}} \int_0^{\theta_i} x F(x)^{n-2} f(x) dx \\
&= \frac{\varphi (n-1)}{\delta(1-\varphi)\bar{v}} \left[\theta_i F(\theta_i)^{n-1} - \int_0^{\theta_i} F(x)^{n-1} dx \right]
\end{aligned}$$

in expectations:

$$E\pi_i = c + E \sum_{j=1}^{n} \pi_i^j$$

where

$$\begin{aligned}
E\pi_i^i &= \frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}} \left[\int_0^1 \theta_i F(\theta_i)^{n-1} f(\theta_i) d\theta_i - \int_0^1 \int_0^{\theta_i} F(x)^{n-1} dx f(\theta_i) d\theta_i \right] \\
E\pi_i^j &= -\frac{1}{n-1} \frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}} \left[\int_0^1 \theta_j F(\theta_j)^{n-1} f(\theta_j) d\theta_j - \int_0^1 \int_0^{\theta_j} F(x)^{n-1} dx f(\theta_j) d\theta_j \right]
\end{aligned}$$

Hence

$$E\pi_i = c$$

We impose symmetry and hence

$$E\pi_i = c = \frac{1}{n}$$

We now need to impose some restrictions on φ for our mechanism to work. This is the case when $0 \leq \pi_i(\theta) \leq 1$ for all i and for all θ. Moreover we impose that $0 \leq \delta < 1$.

In order to do so we need to check the following two inequalities:

$$\begin{aligned}
\pi_i(\theta_i, \theta_{-i}) &\leq 1 \\
\pi_i(\theta_i, \theta_{-i}) &\geq 0
\end{aligned}$$

The first implies that: $\pi_i(1, \mathbf{0}) \leq 1$ and is translated into:

$$\begin{aligned}
\pi_i(1, \mathbf{0}) &= \frac{1}{n} + \frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}}\left[1 - \int_0^1 F(x)^{n-1}dx\right] \leq 1 \\
& \quad \textit{or} \\
\frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}}\left[1 - \int_0^1 F(x)^{n-1}dx\right] &\leq \frac{n-1}{n} \\
\frac{\varphi}{\delta(1-\varphi)\bar{v}}\left[1 - \int_0^1 F(x)^{n-1}dx\right] &\leq \frac{1}{n} \\
\frac{\varphi\Delta}{\delta(1-\varphi)\bar{v}} &\leq \frac{1}{n} \text{ where } \Delta = \left[1 - \int_0^1 F(x)^{n-1}dx\right] \\
\frac{\varphi}{1-\varphi} &\leq \frac{\delta\bar{v}}{n\Delta} \\
\frac{\varphi}{1-\varphi} &< \frac{\bar{v}}{n\Delta} \text{ since } \delta < 1 \\
\varphi &< (1-\varphi)\frac{\bar{v}}{n\Delta} \\
\varphi\left(1 + \frac{\bar{v}}{n\Delta}\right) &< \frac{\bar{v}}{n\Delta} \\
\varphi &< \frac{\bar{v}}{n\Delta + \bar{v}} = \psi(n)
\end{aligned}$$

7 Concluding Remarks

We have constructed a very simple dynamic mechanism that outperforms the bid rotation scheme proposed by McAfee and McMillan (1992). It is similar in nature to the mechanism of Aoyagi

(2003a) but it displays some noteworthy differences. First, the mechanism allows us to pin down the potential it has as a collusive agreement: we can reduce the gap between the equilibrium static bid rotation payoff and the efficient payoff with one third. Second, the mechanism achieves truthtelling in every period but requires that the good is not always allocated to the bidder with the highest valuation, although the valuations are known. We believe this mechanism can possibly be extended to having more states and as such could provide us an answer whether one can construct an equilibrium that is asymptotically efficienct. We leave this for future research.

References

[1] Abreu, D., Pearce D. and Stacchetti, E, 1990. Toward a Theory of Discounted Repeated Games with Imperfect Monitoring, Econometrica, 58.5, 1041-1063.

[2] Aoyagi, M., 2003, "Bid Rotation and Collusion in Repeated Auctions," Journal of Economic Theory, 112, 79-105.

[3] Aoyagi, M., 2003, "Collusion through Mediated Communication in Repeated Games with Imperfect Private Monitoring," forthcoming in Economic Theory.

[4] Athey, S. and Bagwell, K., 2001. Optimal Collusion with Private Information, Rand Journal of Economics, 32.3, 428-465.

[5] Baldwin, L. H., Marshall R.C., and Richard J-F., 1997, Bidder Collusion at Forest Service Timber Sales, Journal of Political Economy, 105, 657-699.

[6] Blume, A. and Heidhues, P., 2002, Modeling Tacit Collusion in Auctions, mimeo.

[7] Crampton, P., and Schwartz, J., 2002, Collusive Bidding in the FCC Spectrum Auctions", in Contributions to Economic Analysis & Policy, 1:1.

[8] Fudenberg, D., Levine, D. and Maskin, E., 1994. The Folk Theorem with Imperfect Public Information, Econometrica, 62.5, 997-1039.

[9] Hendricks, K. and Porter, R., 1989, Collusion in Auctions, Annales d'Economie et de Statististique, 15-16, 217-230.

[10] Johnson, P. and J. Robert, 1998, Collusion in a Model of Repeated Auctions, mimeo.

[11] McAfee, R. and McMillan, J., 1992, Bidding Rings, AER, 82, 579-599.

[12] Pesendorfer, 2000, A Study of Collusion in First-Price Auctions, RES, 67, 381-411.

[13] Skrzypacz, A. and Hopenhayn, H., 1999. Bidding Rings in Repeated Auctions, Rochester Center for Economic Research, Working Paper 463.

8 Appendix

Proof. Let is show that $v^* > \frac{\bar{v}}{2}$. By integration by parts we get that:

$$v^* = \int_0^1 \theta F(\theta) f(\theta) d\theta = \frac{1}{2}\int_0^1 \theta dF(\theta)^2 = \frac{1}{2} - \frac{1}{2}\int_0^1 F(\theta)^2 d\theta$$

At the same time

$$\frac{1}{2}\bar{v} = \frac{1}{2}\int_0^1 \theta f(\theta) d\theta = \frac{1}{2}\int_0^1 \theta dF(\theta) = \frac{1}{2} - \frac{1}{2}\int_0^1 F(\theta) d\theta$$

And hence

$$v^* - \frac{\bar{v}}{2} = \frac{1}{2}\int_0^1 F(\theta)(1 - F(\theta)) d\theta > 0$$

■

Proposition 5 *Let $\varphi < \frac{1}{3}$, and let $\pi = \frac{1}{2}$, Then there exist $\delta(\varphi) = \frac{2\varphi}{(1-\varphi)} < 1$ such that for all $\delta \geq \delta(\varphi)$ our mechanism is an equilibrium of the game of repeated auctions. The expected payoff of this equilibrium is $\varphi v^* + (1-\varphi)\frac{\bar{v}}{2}$. For $\varphi \to \frac{1}{3}$ we need $\delta \to 1$ and the expected payoff converges to $\frac{1}{3}v^* + \frac{2}{3}\frac{\bar{v}}{2}$.*

Proof. Local incentive compatibility implies

$$\begin{aligned}\text{for bidder one } (i) \;&:\; (1-\delta)\varphi\theta f(\theta) + \delta\pi_i'(\theta)(W^R - W^P) = 0\\ \text{for bidder two } (j) \;&:\; (1-\delta)\varphi\theta f(\theta) - \delta\pi_j'(\theta)(W^R - W^P) = 0\end{aligned}$$

$$\begin{aligned}\pi_i' &= -\frac{(1-\delta)\varphi\theta_i f(\theta_i)}{\delta(W^R - W^P)}\\ \pi_j' &= \frac{(1-\delta)\varphi\theta_j f(\theta_j)}{\delta(W^R - W^P)}\end{aligned}$$

Now let π be defined as

$$\pi(\theta_i, \theta_j) = \pi - \frac{(1-\delta)\varphi}{\delta(W^R - W^P)} \int_0^{\theta_i} \theta f(\theta)d\theta + \frac{(1-\delta)\varphi}{\delta(W^R - W^P)} \int_0^{\theta_j} \theta f(\theta)d\theta \tag{23}$$

then we have that

$$E\pi = \pi$$

but for π to be a probability we need that

$$\frac{(1-\delta)\varphi}{\delta(W^R - W^P)} \int_0^1 \theta f(\theta)d\theta \leq \pi \leq \frac{1}{2} \tag{24}$$

This is equivalent to:

$$\frac{(1-\delta)\varphi}{\delta(W^R - W^P)} E\theta \leq \pi \leq \frac{1}{2}$$

We also have from that

$$W^R - W^P = (1-\delta)(1-\varphi)E\theta \tag{25}$$

And our condition boils down to:

$$\frac{\varphi}{\delta(1-\varphi)} \leq \pi$$

$$\frac{\varphi}{\pi(1-\varphi)} \leq \delta$$

Since $\delta < 1$ we have that

$$\frac{\varphi}{\pi(1-\varphi)} < 1$$

and hence we get that

$$\varphi \leq \delta\pi(1-\varphi)$$

$$\begin{aligned} \varphi &\leq \frac{\delta\pi}{1+\delta\pi} \\ \delta &< 1 \\ \pi &\leq \frac{1}{2} \end{aligned}$$

or

$$\varphi < \frac{1}{3}$$

■

Proposition 6 *Let* $M = \{m(\varphi), \pi\}$ *where* $\varphi < \frac{1}{3}$, *and* π *is defined as above. Then there exist a* $\delta^{NR}(M)$ *such that* M *is immune for off-schedule deviations. Then let* $\delta^* = \max\{\delta^{NR}(M), \delta(\varphi)\}$. *For all* $\delta > \delta^*$ *it is the case that* $M = \{m(\varphi), \pi\}$ *is an equilibrium of the repeated auctions game. The expected payoff of this equilibrium is* $\varphi v^* + (1-\varphi)\frac{\bar{v}}{2}$. *For* $\varphi \to \frac{1}{3}$ *we need* $\delta \to 1$ *and the expected payoff converges to* $\frac{1}{3}v^* + \frac{2}{3}\frac{\bar{v}}{2}$.

Proof. Off schedule deviations are deterred by Nash Reversion. The highest incentive to deviate is when a bidder is told not to bid while having the highest valuation (1) . Deviating is then deterred when:

$$\begin{aligned}
(1-\delta)+\delta v^N &< \delta[\varphi v^* + (1-\varphi)\frac{\bar{v}}{2}] \\
\frac{1}{\varphi v^* + (1-\varphi)\frac{\bar{v}}{2} - v^N + 1} &< \delta^{NR}
\end{aligned}$$

The expected payoff of $M = \{m(\varphi), \pi\}$ in each state is:

$$W^R = (1-\frac{2\varphi}{(1-\varphi)})(\varphi v^* + (1-\varphi)E\theta) + \frac{2\varphi}{(1-\varphi)}(\frac{W^P+W^R}{2}) \tag{26}$$

$$W^P = (1-\frac{2\varphi}{(1-\varphi)})\varphi v^* + \frac{2\varphi}{(1-\varphi)}(\frac{W^P+W^R}{2}) \tag{27}$$

Before the auction one randomizes (50/50) who will start in the punishment and reward phase. The expected payoff of the mechanism then is:

$$\begin{aligned}
\frac{W^P+W^R}{2} &= \frac{(1-\frac{2\varphi}{(1-\varphi)})(2\varphi v^* + (1-\varphi)E\theta)}{2} + \frac{2\varphi}{(1-\varphi)}(\frac{W^P+W^R}{2}) \\
\frac{W^P+W^R}{2} &= \frac{(1-\frac{2\varphi}{(1-\varphi)})(2\varphi v^* + (1-\varphi)E\theta)}{2(1-\frac{2\varphi}{(1-\varphi)})} \\
&= \varphi v^* + \frac{(1-\varphi)E\theta}{2} \\
&= \varphi v^* + (1-\varphi)\frac{\bar{v}}{2}
\end{aligned}$$

Hence when bidders become very patient the expected payoff is approaching

$$\frac{1}{3}v^* + \frac{2}{3}\cdot\frac{\bar{v}}{2}$$

■

$$\begin{aligned}
\pi_i(1,\mathbf{0}) &= \frac{1}{n}+\frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}}\left[1-\int_0^1 F(x)^{n-1}dx\right] \leq 1 \\
&\quad \textit{or} \\
\frac{\varphi(n-1)}{\delta(1-\varphi)\bar{v}}\left[1-\int_0^1 F(x)^{n-1}dx\right] &\leq \frac{n-1}{n} \\
\frac{\varphi}{\delta(1-\varphi)\bar{v}}\left[1-\int_0^1 F(x)^{n-1}dx\right] &\leq \frac{1}{n} \\
\frac{\varphi\Delta}{\delta(1-\varphi)\bar{v}} &\leq \frac{1}{n} \text{ where } \Delta=\left[1-\int_0^1 F(x)^{n-1}dx\right] \\
\frac{\varphi}{1-\varphi} &\leq \frac{\delta\bar{v}}{n\Delta} \\
\frac{\varphi}{1-\varphi} &< \frac{\bar{v}}{n\Delta} \text{ since } \delta<1 \\
\varphi &< (1-\varphi)\frac{\bar{v}}{n\Delta} \\
\varphi\left(1+\frac{\bar{v}}{n\Delta}\right) &< \frac{\bar{v}}{n\Delta} \\
\varphi &< \frac{\bar{v}}{n\Delta+\bar{v}}=\psi(n)
\end{aligned}$$

Note that when $n=2$,then $\Delta=\bar{v}$ and hence $\psi(n)=\frac{1}{3}$. For $n>2:\psi(n)>\psi(n+1)$.