Exchange of indivisible goods and indifferences: the Top Trading Absorbing Sets mechanisms*

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Abstract

There is a wide range of economic problems that involve the exchange of indivisible goods with no monetary transfers, starting with the housing market model of the seminal paper by Shapley and Scarf (1974) and including other problems such as kidney exchange or school choice problems. The classical solution to many of these models is to apply an algorithm/mechanism called Top Trading Cycles, attributed to David Gale, which satisfies good properties for the case of strict preferences. In this paper, we propose a family of mechanisms, called Top Trading Absorbing Sets mechanisms, which generalizes the Top Trading Cycles to the general case in which individuals are allowed to report indifferences, while preserving all its desirable properties.

JEL classification: C71; C78; D71; D78

Keywords: housing market; indifferences; top trading cycles; absorbing sets

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1 Introduction

Consider the economy modeled by Shapley and Scarf (1974) in which there is a set of agents, each of whom has strict preferences over a set of indivisible goods, such as houses. In such an economy, commonly known as a "housing market", agents are endowed with one house each and are allowed to swap houses among themselves, although monetary transfers are not permitted. In their seminal paper, using a so-called Top Trading Cycles Algorithm (hereafter, TTC), attributed to David Gale, Shapley and Scarf prove that this model economy has a non-empty strict core.

The housing market has been extensively analyzed in the literature under the domain of strict preferences. It has become plain that the TTC mechanism satisfies very desirable properties. Roth and Postlewaite (1977) prove that this mechanism results in the unique assignment that belongs to the strict core. Subsequently, Roth (1982) shows that it is a dominant strategy for agents to reveal their true preferences. Furthermore, Ma (1994) shows that the TTC mechanism, equivalent to the strict core mechanism, is the only mechanism that satisfies individual rationality, Pareto-efficiency and strategy-proofness (in the domain of strict preferences).

Under the full preference domain, in contrast, very few papers have been written on the housing market, even though it seems quite natural for agents to be indifferent towards goods. One possible reason for this is that the introduction of weak preferences to the model introduces additional complications. First of all, in this case, the strict core might be empty, unique or multi-valued. Moreover, although the core is always non-empty, some of its allocations might be inefficient. As far as we know, there are two papers dealing with weak preferences. The first is by Quint and Wako (2004) and proposes an algorithm to determine whether the strict core is empty or not, and obtain a strict core assignment if it is non-empty. Nevertheless, this cannot be considered a mechanism, since, for housing markets with an empty strict core, it reports that the strict core is empty but does not give an allocation. The other, by Yilmaz (2009), presents a random mechanism satisfying individual rationality, ex-ante efficiency and no justified-envy. However, this is not an strict core mechanism (i.e., there are housing market problems with a non-empty strict core in which the mechanism does not select a strict core allocation). Nor is it a generalization of the TTC mechanism, since the allocation it proposes for problems with strict preferences may be different from the unique strict core allocation. Furthermore, this mechanism does not satisfy strategy-proofness, although it attains higher levels of efficiency.

The contribution of this paper is to present a family of mechanisms that generalizes the TTC mechanism while preserving their good properties when agents are allowed to report indifferences. In order to introduce this family of mechanisms, we define an algorithm called the Top Trading Absorbing Sets algorithm (hereafter TTAS), which results in a strict core allocation when this set is non-empty and a Pareto-efficient core allocation otherwise. Then, we prove that this family of mechanisms satisfies individual rationality, Pareto-efficiency and strategy-proofness.

The literature also describes other problems in which there are indivisible goods and monetary transfers are not allowed. Some examples are housing allocation with existing tenants (Abdulkadiroglu and Sonmez (1999)), kidney exchange (Roth et al. (2004)) and school choice (Abdulkadiroglu and Sonmez (2003)). In the cited problems, the unique solution, or among the proposed solutions, for such problems is one that is based on an adaptation of the TTC to these frameworks. However, as in the housing market problem, they only address the case in which agents have strict preferences. Our family of mechanisms (with the same particular adaptations needed for each framework) will generalize all the classical mechanisms to the case in which agents are allowed to report indifferences.

The rest of the paper is organized into the following sections. Section 2 contains some basic preliminaries of the housing market problem. Section 3 revises the TTC mechanism, introduces the family of TTAS mechanisms and studies the properties of this family of mechanisms in the housing market problem. Section 4 presents some further applications of our mechanisms to other problems. Finally, the proofs of the results obtained throughout the paper are given in the appendix.

2 The housing market model

Let N be a finite set of agents and H be a set of houses such that $|N| = |H| = n$. Each individual $i \in N$ has a transitive and complete (but not necessarily antisymmetric) preference binary relation R_i on H. As usual, P_i and I_i will be used to denote the symmetric and asymmetric parts of R_i, respectively. For any R_i and any $S \subseteq H$, we will define the maximal elements of S according to R_i as the set $max(R_i) = \{x \in S \mid xR_iy \text{ for all } y \in S\}$. Define $R = (R_i)_{i \in N}$. Given $i \in N$, let $R_{-i} = (R_j)_{j \in N \setminus \{i\}}$ denote the preferences of all individuals except i.

An *assignment* (or *allocation*) is a bijective map $\mu : N \longrightarrow H$. In some cases, we will denote the house that is assigned to individual i by μ_i instead of $\mu(i)$. The assignment which describes the initial owners of the houses is called the "initial endowment" and is denoted by ω. For any $T \subseteq N$, we define $\omega(T) = \{x \in H \mid x = \omega_i \text{ for some } i \in T\}$. Then, a housing market is a list (N, H, ω, R).

A *deterministic mechanism* f is a map that assigns for each housing market (N, H, ω, R) an assignment $f(N, H, \omega, R)$. When the description of (N, H, ω, R) is clear, we will denote the house assigned by the mechanism f to individual $i \in N$ as f_i. Let $\mathcal{F}$ be the set of all deterministic mechanisms. A *random mechanism* g is a probability distribution over $\mathcal{F}$. That is, a random mechanism associates to each housing market a probability distribution over the set of assignments. Obviously, any deterministic mechanism is a random mechanism.

An assignment μ is *individually rational* if, for each agent $i \in N$, $\mu_i R_i \omega_i$. A deterministic mechanism f is *individually rational* if it always selects an individually rational assignment for each housing market. A random mechanism is *individually rational* if its support contains only individually rational deterministic mechanisms.

An assignment μ is *Pareto-efficient* if there exists no other assignment ν such that, for all $i \in N$, $\nu_i R_i \mu_i$ and for some $j \in N$, $\nu_j P_j \mu_j$. A deterministic mechanism f is *Pareto-efficient* if it always selects a Pareto-efficient assignment for each housing market. A random mechanism is *ex-post efficient* if its support contains only efficient deterministic mechanisms. A random mechanism g

stochastically dominates another random mechanism h if, for any possible vector of utilities $U = (u_i)_{i \in N}$ compatible with R, the following must hold: for all $i \in N$,

$$\sum_{x \in H} p(g_i(N, H, \omega, R) = x) \cdot u_i(x) \geq \sum_{x \in H} p(h_i(N, H, \omega, R) = x) \cdot u_i(x) \text{ and}$$
there is some $j \in N$ in which this inequality is strict.

Then, a random mechanism g is *ex-ante efficient* if it is not stochastically dominated by any other random mechanism.

A random mechanism g is *strategy-proof* if truth-telling is a dominant strategy in its associated preference revelation game. That is, for any possible vector of utilities $U = (u_i)_{i \in N}$ compatible with R, the following must hold: for all $i \in N$,

$$\sum_{x \in H} p(g_i(N, H, \omega, R) = x) \cdot u_i(x) \geq \sum_{x \in H} p(g_i(N, H, \omega, (R_{-i}, R'_i)) = x) \cdot u_i(x)$$
for all possible R'_i

An assignment μ is in the *core* of the housing market if there is no coalition $T \subseteq N$ and matching ν such that, for all $i \in T$, $\nu_i \in \omega(T)$ and $\nu_i P_i \mu_i$. An assignment μ is in the *strict core* of the housing market if there is no coalition $T \subseteq N$ and matching ν such that, for all $i \in T$, $\nu_i \in \omega(T)$ and $\nu_i R_i \mu_i$ and for some $j \in T$, $\nu_j P_j \mu_j$.

Preliminaries in digraphs

A *directed graph*, or digraph, is a pair (V, E), where V is a set of vertices (or nodes) and E is a set of directed arcs. The *indegree* (*outdegree*) of a node $v_i \in V$ is the number of arcs leading to (leading from) v_i. Given two nodes $v_i, v_j \in V$, we say that there is a *path* from v_i to v_j if there is a sequence of nodes $v_i = v_1, \ldots, v_m = v_j$ such that for all $i \in \{1, .., m-1\}$, there is an arc from v_i to v_{i+1}. A *cycle* is an ordered set of nodes $C = \{v_1, v_2, ..., v_m\}$ such that for all $i \in \{1, \ldots, m-1\}$, there is an arc from v_i to v_{i+1} and there is an arc from v_m to v_1. Two nodes $v_i, v_j \in V$ constitute a *symmetric pair* if there is an arc from v_i to v_j and an arc from v_j to v_i.

An *absorbing set* is a set of nodes A that satisfies two conditions: (i) for any two nodes $v_i, v_j \in A$, there is a path from one to the other (inside connection), and (ii) there is no path from any node $v_i \in A$ to any node $v_j \notin A$ (no inside-outside connection). An absorbing set is *paired-symmetric* if each of its nodes belongs to a symmetric pair.

3 Mechanisms

The classical framework in which the housing market problem is studied in the literature consists of individuals having strict preferences. Shapley and Scarf (1974) have shown that, in this case, the strict core always exists and have proposed the Strict Core mechanism, which selects the unique strict core assignment for each housing market.[1] It has been shown (Roth (1982)) that this deterministic mechanism is strategy-proof. Moreover, Ma (1994) shows that this is the unique mechanism that satisfies individual rationality, Pareto-efficiency and strategy-proofness in this domain of strict preferences. Shapley and Scarf attributed to Gale an algorithm called Top Trading Cycles to compute the strict core assignment of a housing market.

The Top Trading Cycles mechanism

Consider a directed graph in which there are two types of nodes (agents and houses) and arcs leading from agents to houses and from houses to agents, and all nodes have outdegree equal to 1. An interesting fact about any directed graph with these characteristics is that it always has at least one cycle and no two cycles intersect. This allows that the following algorithm, called Top Trading Cycles, always determines an assignment.

Gale´s Top Trading Cycles (TTC) algorithm:

Step 1:

(1.1) Let each agent point to her maximal house and each house point to its owner. Select the cycles of this graph.

(1.2) The agents of the cycles are removed from the algorithm by assigning to each agent the house she is pointing to.

[1] It has been proven by Roth and Postlewaite (1977) that the strict core assignment is unique for housing markets with strict preferences.

Step i:

(i.1) Let each remaining agent point to her maximal house among those remaining and each remaining house point to its owner (note that when an agent leaves, her original house also leaves, so a house remaining in the algorithm implies that its owner also remains in the algorithm and vice versa). Select the cycles of this graph.

(i.2) The agents of the cycles are removed from the algorithm by assigning to each agent the house she is pointing to.

In the general case, in which indifferences are allowed, the strict core may be empty. There is an algorithm (*Top Trading Segmentation*) proposed by Quint and Wako (2004) that determines whether or not a housing market problem has an empty strict core and, in the event of its having a non-empty strict core, finds an allocation of it. Given that all the allocations of the strict core are indifferent for all individuals (i.e., if μ and ρ belong to the strict core, $\mu_i I_i \rho_i$ for all $i \in N$), this algorithm provides a good solution for the case in which the strict core is non-empty. However, there is no satisfactory mechanism that works in all housing market problems, independently of whether or not it has a non-empty strict core. Normally, the mechanism suggested in the literature to generalize the TTC mechanism (see, for example, Roth (1982)) is the following:[2] (1) Take the preferences of those agents who have indifferences and convert them into strict orders by means of (fixed or random) tie-breakers; and (2) apply the Top Trading Cycles mechanism.

It is obvious that this class of mechanisms coincides with TTC for the case of strict preferences. However, in the case of indifferences, the application of these mechanisms does not necessarily lead to efficient allocations. There are, in fact, cases in which these mechanisms never achieve an efficient allocation, independently of the choice of tie-breakers. We illustrate this with a simple example.

Example 1 *Let* $N = \{a_1, a_2, a_3, a_4, a_5\}$ *and* $H = \{h_1, h_2, h_3, h_4, h_5\}$ *be the set of agents and houses. Let* $\omega(a_i) = h_i$ *for all* $i \in \{1, \ldots, 9\}$ *be the initial*

[2] Yilmaz (2009) proposes another mechanism, but it is not a generalization of the TTC mechanism.

endowment. The preference profile is the following:

a_1	a_2	a_3	a_4	a_5
h_2	h_3	h_4, h_5	h_1	h_2
h_1	h_2	h_3	h_5	h_4
h_3	h_1	h_1	h_4	h_5
h_4	h_4	h_2	h_2	h_1
h_5	h_5		h_3	h_3

In this housing market problem, the strict core is empty and the core contains the following four allocations: $\mu^1 = (\mu^1(a_1), \mu^1(a_2), \mu^1(a_3), \mu^1(a_4), \mu^1(a_5)) = (h_2, h_3, h_4, h_1, h_5)$, $\mu^2 = (\mu^2(a_1), \mu^2(a_2), \mu^2(a_3), \mu^2(a_4), \mu^2(a_5)) = (h_1, h_3, h_5, h_4, h_2)$, $\mu^3 = (\mu^3(a_1), \mu^3(a_2), \mu^3(a_3), \mu^3(a_4), \mu^3(a_5)) = (h_2, h_3, h_5, h_1, h_4)$ *and* $\mu^4 = (\mu^4(a_1), \mu^4(a_2), \mu^4(a_3), \mu^4(a_4), \mu^4(a_5)) = (h_1, h_3, h_4, h_5, h_2)$*. There is only one indifference binary relation in the preference profile and, then, there are two possible results of the class of mechanisms presented above, namely* μ^1 *and* μ^2*. However, it is easy to see that these allocations are Pareto dominated by* μ^3 *and* μ^4*, respectively.*

Then, we have that, (i) on the one hand, the TTC mechanism performs well for strict preferences, but the application of tie-breakers is not a good solution for the general case; and (ii) on the other hand, the Top Trading Segmentation algorithm provides a solution for some cases in the general case, but it is not a mechanism in the sense that it provides no allocation when the strict core is empty. In what follows, we propose, for the general case, a family of mechanisms (called Top Trading Absorbing Sets) that extends TTC and TTS and satisfies ex-post efficiency without renouncing to the desirable properties satisfied by TTC.[3]

Top Trading Absorbing Sets mechanisms

To introduce the algorithm that determines the family of mechanisms presented below, we consider directed graphs in which there are two types of nodes (agents and houses) with arcs leading from agents to houses and from houses to agents, all nodes having strictly positive outdegree. An interesting characteristic of

[3]There could be other possible ex-post efficient mechanisms that always select core allocations (and strict core allocations if they exist). However, many of them are not strategy-proof.

these digraphs is that they always have at least one absorbing set (see Kalai and Schmeidler (1977)).

The Top Trading Absorbing Sets (TTAS) algorithm:

Step 0: Consider a priority ranking of the houses; i. e., a complete, transitive and antisymmetric binary relation over H.

Step 1:

(1.1) Let each agent point to her maximal houses and each house point to its owner. Select the absorbing sets of this digraph.

(1.2) Consider the paired-symmetric absorbing sets. Their agents are removed from the algorithm by assigning them their current endowments (Obviously, these houses are also removed).

(1.3) Consider the remaining absorbing sets. For each agent select a unique house to point to, using the following criterion: she points to the maximal house with the highest priority, distinct from her initial endowment.

(1.4) Then, in this subgraph, there is necessarily at least one cycle and no two cycles intersect. Assign (*temporarily*) to each agent in these cycles the house that she is pointing to, but keeps them in the algorithm.

Step i:

(i.1) Let each remaining agent point to her maximal houses among those remaining. Select the absorbing sets of this digraph.

(i.2) Consider the paired-symmetric absorbing sets. Their agents are removed from the algorithm by assigning them their current endowments (Obviously, these houses are also removed).

(i.3) Consider the remaining absorbing sets. Select a unique house for each agent to point to, using the following criterion: she points to the maximal house with the highest priority from those that have not yet been assigned to her.[4]

(i.4) Then, in this subgraph, there is necessarily at least one cycle and no two cycles intersect. Assign (*temporarily*) to each agent in these cycles the house that she is pointing to, but keeps them in the algorithm.

[4] We understand that the initial endowment of an agent is always one of its previously assigned houses.

The following example illustrates how the TTAS algorithm works for a particular housing market problem.

Example 2 *Consider a housing market with* $N = \{a_1, a_2, ..., a_9\}$ *and* $H = \{h_1, h_2, ..., h_9\}$ *and assume that the initial endowment of agent* a_i *is house* h_i *for all* $i \in \{1, 2, \ldots, 9\}$*. Let the preference profile* R *be as follows (we only include the houses that are no worse than the initial endowment of each agent):*

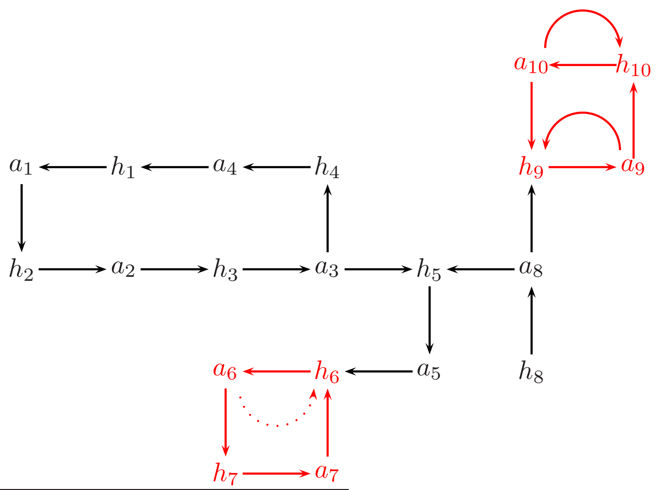
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
h_2	h_3	h_4, h_5	h_1	h_6	h_6, h_7	h_6	h_5, h_9	h_9, h_{10}	h_9, h_{10}
			h_5	h_2					
				h_4					
				h_5					

Consider the following priority ranking of houses:

$$h_1 \succ h_2 \succ h_3 \succ h_4 \succ h_5 \succ h_6 \succ h_7 \succ h_8 \succ h_9 \succ h_{10}$$

In what follows we depict the directed graphs that are formed at each step of the algorithm: [5]

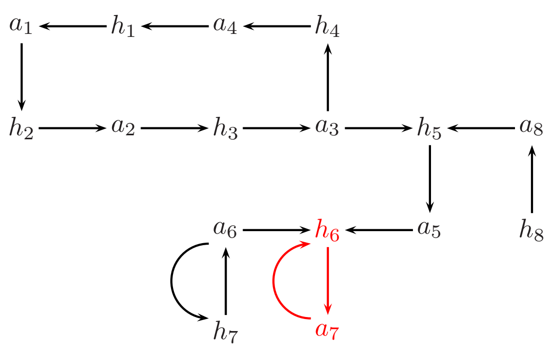
Step 1:



[5]There are two colors for the arrows in each graph: black shows arrows that do not join two nodes of an absorbing set; and red shows those that do. Within the set of read arrows, there are two types: those drawn with a dotted line are the arrows that are not selected by the priority criterion in step (i.3); and those drawn with a continuous line are the ones chosen by the priority criterion.

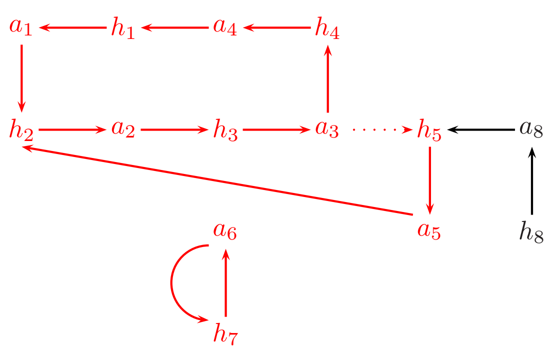
There are two absorbing sets: $A_1^* = \{a_9, h_9, a_{10}, h_{10}\}$, which is paired - symmetric and, hence, is removed by assigning h_9 to a_9 and h_{10} to a_{10}. The other absorbing set is $A_2 = \{a_7, h_7, a_6, h_6\}$. In this case, the priority ranking over houses is applied, and the cycle $c_2 = (a_6, h_7, a_7, h_6)$ is formed. Then, the algorithm temporarily assigns h_7 to a_6 and h_6 to a_7.

Step 2:



There is only one absorbing set: $A_3^* = \{h_6, a_7\}$, which is paired-symmetric. It is removed by assigning h_6 to a_7.

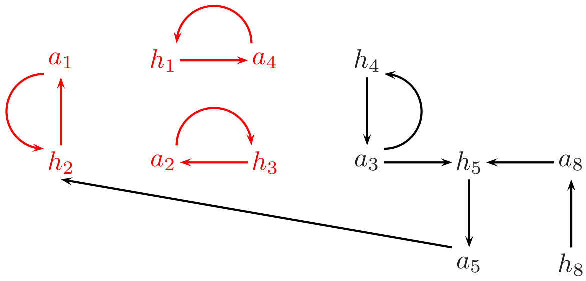
Step 3:



There is a paired-symmetric absorbing set $A_4^* = \{a_6, h_7\}$, which is removed by assigning h_7 to a_6. There is also another absorbing set $A_5 = \{a_1, h_1, a_2, h_2, a_3, h_3, a_4, h_4, a_5, h_5\}$. By applying the priority ranking, the cycle $c_5 = (a_1, h_2, a_2, h_3,$

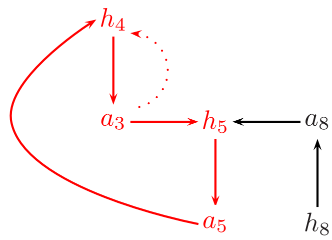
a_3, h_4, a_4, h_1) is formed. Then, the algorithm temporarily assigns h_2 to a_1, h_3 to a_2, h_4 to a_3 and h_1 to a_4.

Step 4:



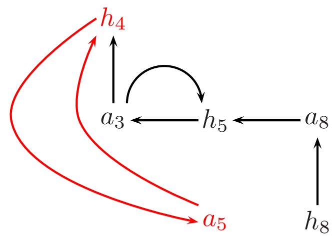
There are 3 paired-symmetric absorbing sets $A_6^* = \{a_1, h_2\}, A_7^* = \{a_2, h_3\}$ and $A_8^* = \{a_4, h_1\}$, which are removed by assigning h_2 to a_1, h_3 to a_2 and h_1 to a_4, respectively.

Step 5:



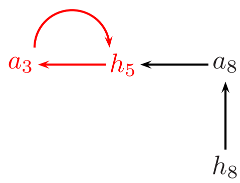
There is only one absorbing set $A_9^* = \{a_3, h_4, a_5, h_5\}$. In this case, the cycle $c_9 = (a_3, h_5, a_5, h_4)$ is obtained by applying the priority ranking. Then, the algorithm temporarily assigns h_5 to a_3 and h_4 to a_5, respectively.

Step 6:



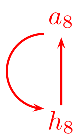
There is a paired-symmetric absorbing set $A^*_{10} = \{a_5, h_4\}$, *which the algorithm removes by assigning* h_4 *to* a_5.

Step 7:



There is a paired-symmetric absorbing set $A^*_{11} = \{a_3, h_5\}$, *which the algorithm removes by assigning* h_5 *to* a_3.

Step 8:



There is a paired-symmetric absorbing set $A^*_{11} = \{a_8, h_8\}$, *which the algorithm removes by assigning* h_8 *to* a_8.

Meanwhile in the description of the TTC is clear that the algorithm always determines an allocation, it is not so clear that this occurs with the TTAS. We prove that the TTAS always selects an allocation in the following proposition.

Proposition 1 *The TTAS algorithm always selects an allocation.*

It is easy to see that in the case in which all individuals have strict preferences, all absorbing sets that appear in any step i of the algorithm are cycles and that, when the trading between agents in each cycle is done, each agent forms a paired-symmetric absorbing set with her new house in step $i+1$ and leaves the algorithm. As a consequence, the TTAS coincides with the TTC when the preferences are strict.

The Top Trading Absorbing Sets algorithm determines an allocation depending on the priority ranking $\succ$ selected in Step 0. Then, we define a mechanism for each priority ranking as follows: a mechanism f is a Top Trading Absorbing Sets mechanism if there exists a priority ranking $\succ$ such that the mechanism chooses, for each housing market problem, the allocation selected by the Top Trading Absorbing Sets algorithm with this priority ranking. We will denote this mechanism by $\sigma^{\succ}$ The selection of the priority ranking is important only in the case in which the strict core is empty, given that, in the remaining cases, it does not affect the level of welfare attained by each individual. However, if the strict core is empty, the priority ranking indicates which individuals, in the event of conflict, have to be treated better than others (but always maintaining the efficiency of the mechanism and the condition that the allocation must be in the core of the problem). Although the priority ranking is written in terms of houses for the simplicity of the algorithm, its interpretation in terms of agents is easy: one individual, i, has priority over another, j, if the initial endowment of i, $\omega(i)$, has priority over $\omega(j)$.

We will first prove that this family of mechanisms always selects an assignment in the core.

Theorem 2 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism always selects an assignment in the core.*

As a corollary, we can deduce that all TTAS mechanisms satisfy Individual Rationality.

Corollary 3 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism is individually rational.*

Now, we prove that TTAS maintains, in the general case, all the properties that characterize the TTC in the restricted case of strict preferences. We start with Pareto efficiency.

Theorem 4 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism is Pareto-efficient.*

Additionally, we also prove that any mechanism belonging to our family is strategy-proof.

Theorem 5 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism is strategy-proof.*

Then, we have proved that, while in the restricted case of strict preferences the TTC mechanism is the only one that satisfies individual rationality, Pareto efficiency and strategy-proofness (see Ma (1994)), in the general case we have a family of mechanisms that satisfy all these properties. Additionally, we are going to prove that our family of mechanisms always selects a strict core allocation if the strict core is non-empty. That is, our family of mechanisms generalizes the solution of Quint and Wako (2004).

Theorem 6 *With any priority ranking $\succ$, the $TTAS^{\succ}$ mechanism selects a strict core allocation when the strict core is non-empty.*

4 Comments and applications

We have proposed a family of deterministic mechanisms for housing market problems in the general case in which individuals are allowed to report indifferences. This family of mechanisms generalizes the previous proposals of the TTC mechanism and the TTS algorithm, satisfying all the desirable properties possessed by these proposals. The mechanisms of the family differ in the priority ranking implemented to favor some individuals over others in the event of conflict, but always without renouncing the requirements of efficiency, strategy-proofness or disregarding the obligation to select a core allocation.

The selection of the priority ranking may be done in terms of some characteristics of the individuals that are not included in the formal specification of the housing market problem (income, seniority, ...). However, if there is no intuitive way of selecting a priority ranking in a particular problem, it is always possible to randomize it. In this case, independently of the probability distribution over the priority rankings, we have that the random mechanism thus obtained satisfies individual rationality, strategy-proofness and ex-post efficiency[6]. Additionally, it selects a strict core allocation with probability 1 if the strict core is non-empty and, in general, it selects a core allocation with probability 1.

There are many other problems in the literature that can be seen as the exchange of indivisible goods: house allocation with existing tenants (Abdulkadiroglu and Sonmez (1999)), kidney exchange (Roth et al.(2004)), school choice (Abdulkadiroglu and Sonmez (2003)). In all these problems, the proposed solution, maintaining the assumption that individuals can only report strict preferences, is based on adaptations of the TTC algorithm to these particular cases.[7] In these problems, it is also natural that individuals may have indifferences. Therefore, it is also necessary to propose a mechanism for the general case in which they can report these indifferences. We can easily adapt our family of Top Trading Absorbing Sets mechanisms to each of these problems analogously to the way in which the original TTC is adapted to incorporate the particular characteristics of each of these frameworks.[8] Therefore, there is a wide range of problems in which our proposed family of mechanisms can be applied.

APPENDIX

We are going to prove all the results of the paper in the Appendix.

[6]This is a difference with respect to the mechanism proposed by Yilmaz (2009), which is ex-ante efficient.

[7]There is an exception in the school choice problem in which, apart from the solution based on the TTC, Abdulkadiroglu and Sonmez (2003)) also propose another mechanism based on the Gale-Shapley (1962) deferred acceptance algorithm, which is the one that some US city authorities have selected to be applied.

[8]The particular details of the adaptations can be provided upon request.

Proof of proposition 1

By contradiction, suppose that this does not occur. That is, there is a maximal set of individuals $S \subseteq N$ and a maximal set of houses $T \subseteq H$ that are never removed from the algorithm. Consider the algorithm immediately after the other agents and houses have been removed (suppose that this occurs in step i). Then, in this subgraph we know that there is at least one absorbing set. If at least one absorbing set is paired-symmetric, we have a contradiction. If, however, all the absorbing sets are non paired-symmetric, we have that in each absorbing set A_i, there exists a set of nodes $B_i \subseteq A_i$ that do not belong to symmetric pairs. We then proceed with step (i.3) and select one house for each individual to point to, using the priority ranking. Then, we go to step (i.4) and proceed with the provisional trading of the houses in the cycles. The nodes of $A_i \setminus B_i$ will also belong to symmetric pairs in the next period. Each node $v_i \in B_i$ will belong to a symmetric pair in the next period if it enters in a selected cycle. Similarly, if there is no node of B_i in any cycle, it is easy to see that in the next period A_i is also an absorbing set.

Then, the set of nodes that belong to symmetric pairs are never decreasing. If they are at any time increasing, we know that we will finally obtain a paired-symmetric absorbing set and, therefore, some agents and houses leave the algorithm and we reach a contradiction. Then, the only possibility is that the agents of B_i never enter in a selected cycle. However, we have seen that in this case the absorbing set A_i will stay stable over time. Given that any node of an absorbing set belongs to some cycle and that the house that each individual chooses to point to varies according to the rule of step (i.3), we can deduce that some node of B_i will finally enter a selected cycle. Therefore, we have a contradiction and the proposition is proved.

Proof of Theorem 2

By contradiction, let μ be the assignment selected by applying the TTAS algorithm with priority ordering $\succ$ to some housing market problem (N, H, ω, R) and assume that μ is not in the core. Then, there exists a coalition $T \subseteq N$ and an assignment υ such that for all $i \in T$, $\upsilon_i \in \omega(T)$ and $\upsilon_i P_i \mu_i$. Denote, without loss of generality, $T = \{1, 2, \ldots, r\}$ such that $\upsilon_i = \omega_{i+1}$ for all $i \in \{1, \ldots, r-1\}$ and $\upsilon_r = \omega_1$. Take $1 \in T$. Given that $\upsilon_1 P_1 \mu_1$, we have that υ_1 left the algorithm before 1. Then, $\omega^{-1}(\upsilon_1) = 2 \in T$ entered a cycle and received a temporary assignment before 1. Moreover, 2 prefers υ_2 to μ_2, which means that υ_2 left the algorithm before 2. Then, $\omega^{-1}(\upsilon_2) = 3 \in T$ entered a selected cycle and received a temporary assignment before 2. Following this argument, we have that for all $i \in \{1, \ldots, r-1\}$, $i+1$ entered a selected cycle and received a temporary assignment before i and, therefore, r entered a selected cycle and received a temporary assignment before 1. However, given that an individual and her initial endowment enter a selected cycle for the first time in the same step, ω_1 is in the algorithm when r first enters a selected cycle. Therefore, the house temporarily allocated to r is no worse than $\upsilon_r = \omega_1$. Then, by Lemma 3, we can conclude that $\mu_r R_r \upsilon_r$, which is a contradiction. And so the proposition is proved.

Proof of Theorem 4

By contradiction, suppose that there is some TTAS mechanism that selects for some housing market problem (N, H, ω, R) an assignment μ which is not Pareto efficient. That is, there exists an assignment ν such that, for all $i \in N$, $\nu_i R_i \mu_i$ and for some $j \in N$, $\nu_j P_j \mu_j$. Given the construction of the algorithm, ν_j left the algorithm with the agent $\mu^{-1}(\nu_j)$ before μ_j. This indicates that $\mu^{-1}(\nu_j)$ belonged to a paired-symmetric absorbing set A at that moment. Given that $\mu^{-1}(\nu_j)$ should, by ν, obtain a house that is at least as good for her than ν_j, we have that at least one agent z of A (probably $\mu^{-1}(\nu_j)$) obtained, by ν, a house ν_z that left the algorithm before z.

We can replicate the analysis with the agent $\mu^{-1}(\nu_z)$. This agent should have left the algorithm with a paired-symmetric absorbing set. Then, at least one agent w of this absorbing set obtained, by ν, a house ν_w that left the algorithm before w. However, this process can not be repeated infinitely: if we return continuously to symmetric absorbing sets that have already left, we will arrive at the first paired-symmetric absorbing set and will be able to continue no further. Therefore, we have a contradiction and the proposition is proved.

Proof of Theorem 5

We are going to prove some lemmas that will help us in the proof of the theorem. The first lemma states that an agent will be indifferent towards all the houses assigned to her temporarily by the TTAS algorithm.

Lemma 1 *Let x_i^t be the t-th temporary assignment that the TTAS algorithm assigns to agent i. Then $\forall t$ $x_i^t I_i x_i^{t+1}$.*

Proof. Consider the step of the algorithm in which x_i^{t+1} is assigned to agent i and let x_i^t be agent i's current assignment. Then, by construction of the algorithm, there is a cycle in which x_i^t points to i and i points to x_i^{t+1} and therefore x_i^{t+1} is maximal for i among the houses remaining in the market/algorithm. Hence $x_i^{t+1} R_i x_i^t$.

Now consider the step of the algorithm in which x_i^t is assigned to i. At this step, there is a cycle in which i points to x_i^t and, by construction, x_i^t is maximal for i among the remaining houses. But in this step, house x_i^{t+1} remains on the market and, therefore, $x_i^t R_i x_i^{t+1}$.

Hence if $x_i^{t+1} R_i x_i^t$ and $x_i^t R_i x_i^{t+1}$ we can conclude that $x_i^t I_i x_i^{t+1}$, as desired. ■

Then, we can deduce the following corollary, by which the first house that the TTAS algorithm assigns temporarily to an agent determines the utility that this agent will obtain from her final assignment.

Corollary 7 *Let x and μ_i be the first temporary assignment and the final assignment that the TTAS algorithm assigns to agent i, respectively. Then $x I_i \mu_i$.*

The following lemma will also help us in the proof of the theorem. We will denote hereafter by $\varphi^{\succ}(P_{-i}, P_i)$ the TTAS mechanism with the priority ranking $\succ$ when the reported preferences are (P_{-i}, P_i) and the description of N, H and ω is clear.

Lemma 2 *Let h_k be the first house assigned temporarily to agent a_i by the TTAS algorithm for (R_{-i}, R_i) with priority ranking $\succ$ and let R_i' be any preference such that $\{h \in H \mid h P_i h_k\} = \{h \in H \mid h P_i' h_k\}$. Then,*

- *the set of selected cycles and paired-symmetric absorbing sets prior to the cycle assigning h_k to agent a_i in the algorithm defining $\varphi^{\succ}(R_{-i}, R_i)$ is also in the algorithm defining $\varphi^{\succ}(R_{-i}, R_i')$, and*
- *(ii) Each agent participates in the same sequence of temporal assignments in the algorithm defining $\varphi^{\succ}(R_{-i}, R_i)$ until agent a_i is assigned h_k as in the first v steps of the algorithm defining $\varphi^{\succ}(R_{-i}, R_i')$.*

Proof. Consider that h_k and a_i enter a selected selected cycle of the algorithm defining $\varphi^{\succ}(R_{-i}, R_i)$ in step q.

Let $t = 1$ be the first step of the algorithm and let $G_1(R_{-i}, R_i)$ be the graph associated with this step when agent a_i declares R_i. Suppose that $q > 1$ (if not, the proof is finished). Notice that the paired-symmetric absorbing sets in $G_1^{\succ}(R_{-i}, R_i)$ (the digraph in step 1) are also in $G_1^{\succ}(R_{-i}, R_i')$. Let $\mathcal{C}_\infty$ denote the set of cycles obtained by the algorithm at the end of this step and let $c_j = \{a_1, h_2, a_2, h_3, ..., h_1\}$ be a cycle in $\mathcal{C}_\infty$. Now consider $G_1^{\succ}(R_{-i}, R_i')$. Notice that every agent in c_j is in this graph pointing to the same houses as in $G_1^{\succ}(R_{-i}, R_i)$ (given that $a_i \notin c_j$). (a) If c_j is in an absorbing set in $G_1^{\succ}(R_{-i}, R_i')$, then the same structure of priorities is used to select an arrow from each agent of c_j in $G_1^{\succ}(R_{-i}, R_i)$ and in $G_1^{\succ}(R_{-i}, R_i')$. Hence c_j is also obtained as a cycle in the first step of the algorithm for (R_{-i}, R_i'). (b) If not, all agents and houses in c_j will first enter an absorbing set (the same absorbing set for them all) in the same step of the algorithm. Then, in the step in question, say step t, the structure of priorities gives the same result as in $G_1^{\succ}(R_{-i}, R_i)$ and then the cycle c_j is also obtained in $G_t^{\succ}(R_{-i}, R_i')$ and this is the first cycle in which agents in c_i enter.

Consider now $t = 2$ (assume that $q > 2$, if not, the proof is finished) and let $G_2^{\succ}(R_{-i}, R_i)$ be the graph associated with this step when agent a_i declares R_i. Consider any paired-symmetric absorbing set, A_i, in this graph. It is easy to verify that every arrow from an agent in A_i to any house outside A_i in $G_1^{\succ}(R_{-i}, R_i)$ is not in $G_2^{\succ}(R_{-i}, R_i)$. This happens because these houses belong to a paired-symmetric absorbing set in $G_1^{\succ}(R_{-i}, R_i)$. Then, every arrow from an agent in A_i in $G_2^{\succ}(R_{-i}, R_i)$ is the same as in $G_2^{\succ}(R_{-i}, R_i')$ and in the subsequent steps (given that the paired-symmetric absorbing sets in $G_1^{\succ}(R_{-i}, R_i)$ and in $G_1^{\succ}(R_{-i}, R_i')$ are the same). Consider now a house $h_i \in A_i$. If h_i points to its original owner in A_i, it also points to her in $G_2^{\succ}(R_{-i}, R_i')$ in the subsequent steps. If h_i points to an agent a_j different from its original owner, it must belong to a cycle c_j obtained in $G_1^{\succ}(R_{-i}, R_i)$. Then, by the previous reasoning, we know that c_j will also be obtained in $G_t^{\succ}(R_{-i}, R_i')$ for some t. Therefore, we obtain that the paired-symmetric absorbing sets A_i will be obtained in $G_{t^*+1}^{\succ}(R_{-i}, R_i')$ (where t^* is the later step at which a house in A_i has entered its corresponding selected cycle[9]). And, therefore, the sequence of temporal assignments that has received each of these agents are the same in both cases.

[9]If all of them point to their original owner, $t^* = 1$

Consider now a cycle $c_j = \{a_1, h_2, a_2, h_3, ..., h_1\}$ in $G_2^{\succ}(R_{-1}, R_i)$. In c_j there may be (only) three types of agents: (i) Those agents that do not enter a selected cycle in the first step in $G_1^{\succ}(R_{-i}, R_i)$ and point to houses that are not present in $G_2^{\succ}(R_{-i}, R_i')$. (ii) Those agents that do not enter a selected cycle in the first step and do not belong to (i). (iii) Those agents that enter a selected cycle in the first step. The houses that disappear in the first step are the same in $G_1^{\succ}(R_{-i}, R_i)$ and in $G_1^{\succ}(R_{-i}, R_i')$ because the paired-symmetric absorbing sets are the same in both graphs (as proven above). Therefore, the agents in (i) point to the same houses in $G_2^{\succ}(R_{-i}, R_i)$ and in $G_2^{\succ}(R_{-i}, R_i')$ (and probably in the subsequent steps). By triviality, the agents in (ii) also point to the same houses in $G_2^{\succ}(R_{-i}, R_i)$ and in $G_2^{\succ}(R_{-i}, R_i')$. With respect to the agents in (iii), they point to all their maximal houses in $G_2^{\succ}(R_{-i}, R_i)$. We know that the cycle formed by each of them in the step 1 of (R_{-i}, R_i) is also formed at some step t of (R_{-i}, R_i). Let t^* be the later step at which one of the cycles is formed ($t^* = 1$ if this set is empty). With respect to the houses in c_j, it is easy to verify that they point to the same agent in $G_2^{\succ}(R_{-i}, R_i)$ and in $G_{t^*+1}^{\succ}(R_{-i}, R_i')$. Then, $G_2^{\succ}(R_{-i}, R_i)|c_j = G_{t^*+1}^{\succ}(R_{-i}, R_i')|c_j$ [10]. Then we also have that there exists $\hat{t} \geq t^* + 1$ such that $G_2^{\succ}(R_{-i}, R_i)|c_j = G_{t^*+1}^{\succ}(R_{-i}, R_i')|c_j$ and c_j belongs to some absorbing set in both graphs. Given that c_j is formed in $G_2^{\succ}(R_{-i}, R_i)$, we can deduce that the house (h_j) temporarily allocated to each agent (a_j) in c_j at this step is the house with the highest priority between the set of maximal houses among the remaining ones. It is possible that in $G_{\hat{t}}^{\succ}(R_{-i}, R_i')$ set of maximal houses among the remaining ones of each agent is a proper subset of that in $G_2^{\succ}(R_{-i}, R_i)$ but h^{j+1} is still present and, therefore, must be the house with the highest priority. Then c_i is also formed in step $\hat{t}$ and the sequence of temporal assignments to each of these agents is the same in both cases.

When $t \in \{3, \ldots, q\}$, the proof is similar. Therefore, we have proved both parts of the lemma and v will correspond with the maximum of all $\hat{t}$ that will appear in the proof of all steps of the algorithm after a_i and h_k enter a cycle. ■

Now, we prove that, if an individual attains a utility level declaring a preference R_i, then there exists a house that gives this individual the same utility such that if the individual makes it her maximal house, the mechanism will assign it

[10] If G is a graph and c is a set of nodes of G, we denote by $G|c$ the restricted graph that includes only the nodes of c and the arrows leading from a node of c to another node of c.

to her.

Lemma 3 *Let $U_i(\varphi_i^{\succ}(R_{-i}, R_i)) = k$, then there exists a house h_j such that $U_i(h_j) = k$ and $\varphi_i^{\succ}(R_{-i}, R_i') = h_j$ for all R_i' with $\max(R_i') = h_j$.*

Proof. Let h_j be the first house assigned temporarily to agent a_i by the TTAS algorithm for (R_{-i}, R_i) when the priority ranking is $\succ$. Notice that, by Corollary 7, $U_i(h_j) = U_i(\varphi_i^{\succ}(R_{-i}, R_i) = k$. By Lemma 2, we have that the absorbing set of agent a_i in the graph corresponding to step q of the algorithm in which h_j is assigned to a_i when she declares R_i is the same as the absorbing set of agent a_i in the graph corresponding to step v of the algorithm when she declares R_i'. Additionally, we know that each agent has passed from the same sequence of temporal assignments in both algorithms until these steps are reached. Then, if the priority criterion has selected the cycle in which h_j is assigned a_i when she declares R_i, the priority criterion has to select this cycle also when she declares R_i'. Given that $\max(R_i') = h_j$, we have, by Corollary 7, that $\varphi_i^{\succ}(R_{-i}, R_i') = h_j$. ■

Now, we can prove the theorem. By Lemma 3, we have that, if there is any way of obtaining a particular level of utility, the same level can also be obtained by declaring any preference in which the maximal house is one of the houses (h_k) that provides this utility. Consider in particular a ranking R_i' in which h_k is the unique maximal house, $h_k P_i \varphi_i(R_{-i}, R_i)$ and $\{h \in H \mid h P_i h_k\} = \{h \in H \mid h P_i' h_k\}$. Then, by Lemma 2, we have that the set of cycles and paired-symmetric absorbing sets that were formed when she declares R_i before obtaining $\varphi_i^{\succ}(R_{-i}, R_i)$ are also formed when she declares R_i'. Then, in particular, we know that h_k belongs to a paired-symmetric absorbing set and leave the algorithm with an agent different from a_i when a_i declares R_i. Therefore, h_k also left the algorithm with this different agent when a_i declares R_i'. Hence, it is impossible for a_i to obtain a better house than $\varphi_i^{\succ}(R_{-i}, R_i)$ and the theorem is proved.

Proof of Theorem 6

Consider a housing market problem (N, H, ω, R) with a non-empty strict core. We need to introduce an algorithm, called Top Trading Segmentation (hereafter, TTS) originally proposed by Quint and Wako (2004) to determine a partition of the set of agents and houses.

Step 1: Let each agent point to her maximal houses and each house point to its owner. Select the absorbing sets of this digraph. Each absorbing set constitutes an element of the partition.

Step i: Let each agent point to her maximal houses among the remaining ones and each remaining house point to its owner. Select the absorbing sets of this digraph. Each absorbing set constitutes an element of the partition.

With the partition obtained with the algorithm, Quint and Wako (2004) proved that the following statements are equivalent:

- In each element of the partition, it is possible to find a sub-allocation that assigns to each agent one of her maximal houses in this set.
- The strict core of the problem is non-empty and one of its allocation consists of the union of all these sub-allocations.

Now, we will prove the following lemma.

Lemma 4 *Consider a digraph (V, E) such that V is an absorbing set. Then, we can partition V into a set of disjoint cycles if and only if there exists a subset $E' \subseteq E$ such that in the digraph (V, E') all nodes have indegree and outdegree equal to 1.*

Proof. First assume that we can partition V into a set of disjoint cycles, $\{c^i = (h^i_1, a^i_1, h^i_2, a^i_2, \ldots, h^i_{m_i}, a^i_{m_i})\}_{i\in\{1,\ldots,k\}}$. Then, construct E' as follows: $(x, y) \in E'$ if and only if $[x = a^i_j$ and $y = h^i_{j+1}]$ or $[x = h^i_j$ and $y = a^i_j]$ for some $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, m_i\}$. Then, it is easy to see that in (V, E') all nodes have indegree and outdegree equal to 1.

Assume now that there exists a subset $E' \subseteq E$ such that in the digraph (V, E') all nodes have indegree and outdegree equal to 1. Then, start with any node of the digraph as the first node of a cycle. Then, continue with the unique sequence of edges (given that the outdegree of all nodes is 1) leading from this node. Given that the indegree of all nodes is equal to 1, this sequence will at some point terminate at the initial node. Then, this sequence is the first cycle of the partition. Starting with other node that does not belong to this cycle, we will construct another cycle, disjoint from the first one. Finally, following this procedure, we will have a partition of V into a set of disjoint cycles. ■

Then, consider any absorbing set from the first step of the TTAS algorithm. Note that this absorbing set is also one of the sets of the partition determined by TTS. Then, given the result of Quint and Wako (2004), we have that in a housing market problem with a non-empty strict core, the absorbing sets determined in step 1 of the TTAS algorithm must have a partition into disjoint cycles. Or, equivalently, using Lemma 4, in each of these absorbing sets we can find a subset of edges such that the indegree and outdegree of each of these nodes is equal to 1.

If the absorbing set is paired-symmetric, our algorithm gives to each agent in step (1.2) one of her maximal houses. Given Corollary 7, we deduce that our mechanism assigns to these individuals one of their maximal houses.

If the absorbing set is non paired-symmetric, our algorithm applies, in step (1.3), a priority ranking $\succ$ to determine only one edge for each node of the absorbing set, and, in step (1.4), the resulting cycles provisionally trade their houses. If the priority ranking chooses exactly the edges that determine the partition into cycles of the strict core allocation, we will have that all individuals in the absorbing set will provisionally attain one of their maximal houses. Given Corollary 7, we deduce that our mechanism allocates to these individuals one of their maximal houses.

If, however, the priority ranking chooses other different edges, we need to prove that, in the second step of the algorithm, we will have that the same agents belong to absorbing sets such that we can find a subset of edges in which all the associated nodes have in-degree and out-degree equal to 1. Given that this condition is satisfied in step 1, we are going to construct a function that assigns, for each of the edges belonging to this subset in step 1, an edge in step 2. First, for each edge from an agent to a house, consider exactly the same edge in step 2. Second, for the edges from houses to agents, select the unique edge that part from each house in step 2. Now, we will prove that taking into account these edges, the nodes will have indegree and outdegree equal to 1. Given that we have selected only one edge starting from any node, the outdegree is equal to 1 in all nodes. To see why the indegree is also equal to 1, consider first the agents nodes. In each of this nodes, we have that the unique house that points to it is the one that in this moment belongs to the agent, and, hence, the indegree is equal to 1. Consider now the nodes of the houses. Given that we have selected the same edges as in step 1 of the algorithm and that the condition was satisfied in that step, we have that the indegree is here also equal to 1 for any node.

Finally, if the condition is satisfied for all the absorbing sets determined in step 1 of the TTAS algorithm, it is easy to see that the other sets of the partition obtained by the TTS algorithm will appear in subsequent steps of the TTAS algorithm and the same reasoning applies. Therefore, the theorem is proved.

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