A Normative Measure of Fragmentation

Jean-François Caulier *

1 Introduction

Measuring the fragmentation of a party-system has long been recognized as one of the most important characteristics for comparative purposes in politics, the simple number of parties alone being of poor utility. The last decades have seen the success of one such measure growing, supplanting all others because of its very intuitive and ready-to-use interpretation, and has been designed by Laakso and Taagepera [15] in 1979 under the name of Effective Number of Parties. The number we obtain using the formula tells us the number of parties the assembly would have if every party had the same bargaining power in the decision-making process. It is thus an hypothetical situation judged to be equivalent to the existing one, given the distribution of seats across parliamentary parties and the decision rule.

The idea of aggregating in a meaningful measure a distribution of numbers is neither new nor specific to political science. Inequality Indices (see Atkinson [2], Dasgupta, Sen and Starrett [8], Kolm [14]) want to show the degree of fairness -or unfairness, it depends on how you interpret it- of a distribution of wealth among individuals of a population. Industrial Concentration Indices (see e.g. Encaoua and Jacquemin [10], Hannah and Kay [13]) tell us how far a market is dominated by large firms. A highly concentrated industry will take on a higher value of the index and indicates that we are closer to the monopoly end of the competition spectrum. Another related concept is the one of Entropy in Information Theory (see e.g. Rényi [18], Shannon [20], Theil [22]). The concept of entropy is deemed to measure the uncertainty represented by a probability distribution. When a particular state of the world is attained with certainty, no information is thus conveyed and the measure takes on its minimum value.

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Loosely speaking, all these kinds of measure have the same point of departure: they are summary statistics reflecting the information contained in a vector of numbers in a practical and intuitive way.

A Fragmentation Measure follows the same line: it evaluates the extent to which bargaining power is scattered among parties in a political assembly (see e.g. Sartori [19] (more precisely p.120), Laakso and Taagepera [15], Lijpart [16], Caulier and Dumont [6]). Many reasons can be put forward for being interested in the degree of fragmentation of a party-system. Fragmentation has been recognized as one of the most significant characteristics of party-system for comparative purposes. In his seminal work, Blondel [4] was the first to look at both the number and the relative size of parties in order to identify the dominance of a party in a system. Since then, the Effective Number of Parties (ENP below) became the standard approach in order to assess the political power fragmentation. The notion of equivalent-number is rooted in the industrial concentration literature as an inverse measure of concentration. It is only recently that the ENP measure has been criticized for its inadequate and counter-intuitive behaviour under a number of circumstances (see for example Taagepera [21], Dunleavy and Boucek [9]). The most striking misleading result of this index can be observed in single-party majority situations as it still indicates that more than one party is relevant in terms of government formation. If we think about fragmentation as “number-equivalent” parties in term of bargaining power, in such a situation only one party has to be considered as relevant. The index should correctly reflect this case.

In this paper we are only concerned with measurement of fragmentation in representative party-systems. More precisely, our purpose is to discuss the desired properties of a fragmentation index measured as ‘number equivalent’ with the help of the relation it has with industrial concentration and inequality. From now on, when we talk about a fragmentation measure, it will always be in term of equivalent-number.

The primary motive of this paper is to seek how the Effective Number of Parties of Laakso and Taagepera could be improved where it appears to be flawed. First, this could not be done by using existing industrial concentration or inequality indices on the distribution of seats of the political parties in a parliament. We will see the reasons in exploring the desired properties of a fragmentation index and those required in the two other fields. Second, we claim that a first step to improvement consists in replacing the seats shares of political parties by their power indices, which are numerical representation of their relevance.

In line with Caulier and Dumont [6], we claim that a rigorous definition of fragmentation will require a precise explanation of the unit on which it has to be applied: these units are the distribution of power indices calculated from the seat shares of the parliamentary parties. This is the purpose of section 2, where we present the domain on which a fragmentation measure
has to be applied, and we do it in a broader sense than Caulier and Dumont [6] in order to gain in generality in the definition of power index one could use. In section 3, we discuss about the similarities and differences we can observe between inequality, concentration and fragmentation measures. In the section 4, we turn on to the desired properties an index of fragmentation should present and look at their implications. These properties are presented in an axiomatic approach and are coming mainly from the industrial concentration literature, in order to present the characterization of the ENP in term of relevance in a decision-making context. We continue the section 4 by presenting the specificities of fragmentation: we define the notion of contribution to fragmentation of a party and from the domain of definition, the form and the properties of the $cf$ function we define a new measure called *Maximum Contribution Index of Fragmentation*. In section 5 we pass through the desired properties fulfilled by the Maximum Contribution index and show why it could not be used to evaluate inequality or industrial concentration because its violation of main axioms of inequality and concentration. The last section concludes.

2 Unit of Measurement

Any summary measure like concentration or inequality index meets the same two practical problems at the very beginning. The first one deals with the unit of measurement of the numbers contained in the list to be resumed, the second one with the individuals corresponding to each number.

2.1 Fragmentation of Power

In order to properly resume the information contained in a vector of numbers, those numbers have to be correctly defined and collected. For example, in inequality what is meant by ‘wealth’ has to be precisely determined: is it revenue per capita in 1995 dollar, head of family revenue in power parity purchase, etc . . .

Concentration indices refers to the degree of control of economic activity by large firms. But what is the ‘economic activity’ of a firm? One possible measure is the assets of the firm. Another is the number of employed workers, or the value added generated.

For fragmentation, this kind of definitional problem is never questioned: all existing indices use seats (or seat shares) of parties in the calculation. The consequence is that none of the existing measures behaves adequately in the presence of a majoritarian party. Recent researches in the field have thus tried to develop new or additional formulas with the purpose to cope with this problem.1 Despite the problems met in using the existing indices, a large

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1Good examples are the aforementioned Taagepera [21] and Dunleavy [9].
A consensus has been reached by political analysts to use the Effective Number of Parties. All later indices that have been developed are modifications of the ENP in a sense, such as to fit better with the feeling specialists have of the party-system under study. In this article, we want to give an operationalization of the concept of fragmentation, but because what is meant to be measured by a fragmentation index is not always identical among the authors, we will here agree on the definition proposed by Sartori: fragmentation measures the repartition of bargaining power, or coalitional power (see Sartori [19]). We will thus follow the path initiated by Caulier and Dumont [6] and propose to replace seat shares by voting power indices in the computation. As argued by Gallagher et al.,[12], ‘...the ways in which the bargaining power [is distributed] can sometimes differ quite starkly from the distribution of seats in the legislature’ (2001: 344), and indeed the ability to influence a decision is not well represented by a party’s share of parliamentary seats when there are few parties. Thus, in this paper, we argue that a fragmentation measure in term of number of relevant parties has to be calculated on a vector of power indices distribution rather than on a seats distribution. Among the most well-known, or at least the most cited indices we find the Shapley-Shubik and the Banzhaf (Penrose) indices. But our object here is not to discuss on which among the existing power indices better represents the distribution of bargaining power of parties in a parliament. The only thing we require for the distribution of power is to be generated by a consistent power index. What we mean by consistent is presented in the next section where we give a formal and general definition of a power index.

2.2 Power Measurement

The object of this part is to give the reader a clearer view of what is meant by a power index in a political setting. A parliament is a finite set $N$, consisting in $|N|$ parties, $|N| \in \mathbb{N}$, with $\mathbb{N}$ the set of natural numbers, with decision rule $q \in \mathcal{S}$, with $\mathcal{S}$ the set of all possible decision rules, that can be summarized by a pair $(q; (s_i)_{i \in N})$ where $s_i$ is the number of seats of party $i$. A distribution of seats is denoted $(s_i)_{i \in N} \in \mathbb{R}_{0+}^{|N|}$, with $\mathbb{R}_{0+}^{|N|}$ the $|N|$-fold cartesian product of real numbers with the origin deleted.

For example, in a qualified majority system, $q$ is the quota of seat shares parties have to gather in order to implement a decision. Consecutively, we can compute for each party $i$ the number of times it appears to be a swing player for a coalition, that is for all possible coalitions $S \in 2^{|N|}$, if

\footnote{We refer the interested reader to the book of Felsenthal and Machover [11] for a complete discussion on power indices.}
\[ \sum_{i \in S} s_i > q \] then we give the value \( v(S) = 1 \) to the coalition \( S \) otherwise \( v(S) = 0 \).

The number of times party \( i \) is a swing player is computed as:

\[
\eta_i = \sum_{S \in \mathcal{P}(N)} (v(S) - v(S \setminus \{i\}))
\]

**Definition 1.** Party \( i \in N \) is a Dummy if \( \eta_i = 0 \), it is a swing player for none coalition.

**Definition 2.** Party \( i \in N \) is a Dictator if \( \eta_j = 0 \) for all \( j \neq i \), \( i, j \in N \).

**Definition 3.** For an \(|N|\)-party parliament, a Power Index is a real valued function \( \beta \) defined on \( \mathcal{F} \times \mathbb{R}^{|N|}_+ \):

\[
\beta : \mathcal{F} \times \mathbb{R}^{|N|}_+ \rightarrow \mathbb{R}^{|N|}_+
\]

A power index transforms a distribution of seats \( s \in \mathbb{R}^{|N|}_+ \) into a distribution of ‘power’ \( p \in \mathbb{R}^{|N|}_+ \), for a given decision rule \( q \in \mathcal{F} \), with \( p = (p_i)_{i \in N}, p_i \) the power index of party \( i \) and \( \mathbb{R}^{|N|}_+ \) the nonnegative orthant of the Euclidian \(|N|\)-space of real numbers. Note that a party cannot have zero seats in a parliament but can indeed have zero power. In this case the party is called dummy and is neither able to influence a decision inside the parliament. We thus allow for a very broad class of power measurements, including the absolute ones,\(^3\) because what we need is fragmentation formulas that can be used on every situation, whatever the chosen power index used to measure parties relevance.

But usually, only the relative distribution of seats matters in the computation of the power of parties. We could thus apply the power index on a seat share distribution \( \bar{s} = (\bar{s}_i)_{i \in N} \) where \( \bar{s}_i = \frac{s_i}{\sum_{i=1}^{|N|} s_i} \) that belongs to the simplex of order \((|N| - 1): \Delta^{|N| - 1}\). We can thus equivalently have

**Definition 4.** For an \(|N|\)-party parliament, a Power Index is a real valued function \( \beta \) defined on \( \mathcal{F} \times \Delta^{|N|-1} \):

\[
\beta : \mathcal{F} \times \Delta^{|N|-1} \rightarrow \mathbb{R}^{|N|}_+
\]

### 2.3 Individuals of Measurement

Prior any investigation, the individuals on which the number will be collected have to be well-defined. This aspect is more a practical one than a purely desired property. The problem here is to obtain reliable information that are privately held, such as exact amount of revenue for an inequality

\[^3\text{That don’t add up to one.}\]
measure or the number of clients for a concentration index. What concerns fragmentation in politics, this problem should not be neglected. In some countries (like Turkey for example), political parties appear to be very unstable factions with high rotation of politicians between different parties. Hence, once the power of each party has been computed in a parliament, we need each party to act as unitary actor, politicians from the same party remain in the same party during all the legislature and they always agree on how to vote for every decision.

3 Inequality - Concentration - Fragmentation - Equivalent Number: A Panoply

Up to now, no systematic difference has been drawn between e.g. an inequality and a concentration index, or between a concentration and an effective number of components. Even if they share the same logical fundamental feature, all these tools refer to different aspects of life. The most striking distinction between inequality, concentration and fragmentation is the underlying notion of welfare for the two first notions. Inequality index seeks to reflect the extent to which we could attain a better situation for the society, more equal, and industrial concentration the extent to which we are far from a perfect competition situation, and price level is higher than the marginal cost. With fragmentation, the purpose is only descriptive, not prescriptive. The task is to develop an as independent as possible criterion.

The distinction between inequality and concentration has long been noticed by Hannah and Kay [13]. Suppose we were to apply an inequality index to measure the concentration in an industry. The inequality measure will compare the output levels of small firms with those of large firms. In contrast, measures of concentration offer an evaluation of the dominance of firms of significant size. To see clearly the point of distinction take an industry [Hannah and Kay [13] p.50]: ‘dominated by a small number of giants of similar size. Now suppose many small firms enter, but enjoy little success, so that even in the aggregate their market share is very low. Then concentration has not been significantly affected, but degree of inequality in firms’ size has greatly increased’.

For a fixed number of firms, indices of concentration are strictly equivalent to inequality measures. In the comparison of two output vectors (or market shares distributions) with the same degree of inequality, if the number of firms differs, the output configuration with the fewer number of firms is more concentrated.

The relationship between concentration measures and effective number of firms is given by Blackorby et al.[3]: ‘the equivalent number of equal-sized firms - number equivalent for short- has gained considerable currency as an inverse measure of industrial concentration. In constructing a numbers-
equivalent we are to imagine taking the existing industry output and having it produced by firms with equal market shares. The number of such firms is chosen so that this hypothetical state of affairs is judged to be equivalent to the existing distribution of output across firms.’ If you take a standard concentration index, such as Herfindhal (H), to assess concentration in an industry consisting in $|N|$ firms, and you invert it, all firms having the same market share (output share) give the index a value of $|N|$. Following Blackorby et al. [3] we write a number equivalent measure $D$ as

$$D(x) = f(|N|, I^{[N]}(x)),$$

where $x \in \mathbb{R}^{[N]}$ is the distribution of shares, $|N| \in \mathbb{N}$ is the number of elements, $|N| > 1$ and $I$ a measure of inequality. The function $f$ is increasing in $|N|$ and decreasing in $I$.

In assessing political fragmentation, the ENP designed by Laakso and Taagepera [15] in 1979 has become the standard numerical measure for the comparative analysis of party systems. The way they constructed their indice was simply to apply the Herfindhal measure on seat shares distribution of parties, and invert it, so as to obtain a number-equivalent measure. First, because in the construction of the index, more weight is given to the largest parties, bigger parties contribute more to the fragmentation than smaller ones. In the ENP index the shares of seats $^4$ of a party are self-weighting (by squaring these values). The precise calculation is:

$$ENP = \frac{1}{\sum_{i=1}^{n} (s_i)^2} = \frac{1}{H}$$

using the same notation as above.

Second, the ENP takes very intuitive results. Suppose the distribution of seats between three parties in a given parliament is $s = (1/3, 1/3, 1/3)$. In such a case, the ENP is exactly 3.00. When seats are equally distributed among the parties, the ENP coincides with the raw number of parties (the maximum value of the index). This means that doubling the number of equal-sized parties provides an ENP value twice higher. Suppose now that one party gets more seats than the others, the ENP will go down, approaching 2.00. A 3-party system with an ENP lower than 3.00 tells us that some parties are somewhat ‘dominated’ by others. The value given by the ENP can thus be interpreted as the number of hypothetical equal-sized parties. But in evaluating the number of relevant parties competing or being influential for the building of a majority government, the ENP can sometimes lead to very counter-intuitive results, and thus gives a wrong picture of how coalition or blackmail potential is distributed. Because seats are far from synonymous to bargaining strength, Caulier and Dumont [6] proposed to

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$^4$Remember that shares add up to one: $\sum_{i=1}^{[N]} s_i = 1$. 

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improve the Laakso and Taagepera’s index in using the formula on a distribution of power indices rather than on seat shares distribution. Using the same notation as the preceding section:

\[ s = (s_i)_{i \in N} \]

A seat shares distribution for a \(|N|\)-party parliament.

\[ \beta : \mathcal{S} \times \mathbb{R}^{|N|}_{0+} \rightarrow \mathbb{R}^{|N|}_+ \]

a consistent power measure.

\[ p = \beta(s) = (p_i)_{i \in N} \]

A power distribution for a \(|N|\)-party parliament.

The power shares distribution is

\[ \bar{p} = (\bar{p}_i)_{i \in N} \]

where

\[ \bar{p}_i = \frac{p_i}{\sum_{i=1}^{|N|} p_i} \]

The Effective Number of Relevant Parties (ENRP) =

\[ \frac{1}{\sum_{i=1}^{|N|} (\bar{p}_i)^2} \]  \hspace{1cm} (3.3)

This criterion permits to compare the equivalence between two distributions of seats transformed in power. If the ENRP of a parliament is 3, this parliament is said to be equivalent to one with three parties having the same negotiation power. Another parliament with a ENRP of 6 will said to be twice as fragmented then the first one. In the next section, we present the desired properties one could impose to a fragmentation measure to fulfil and will see which of these are met by the ENRP. Subsequently, we present another measure that meet the same properties than the ENRP but one and we argue that fragmentation in politics is better represented by measure that satisfy that collection of axioms.

In the section 5, we will turn back on the comparison of inequality, concentration and fragmentation in equivalent-number measures examining which fundamental properties of inequality and concentration is not met by a fragmentation measure.
4 Axioms of Fragmentation

Because the close connection between equivalent-number measures and concentration indices, most of the axioms presented here are common to the literature of the latter. We begin by a formal definition of a fragmentation index as number-equivalent:

Consider the set of all parties around the world in any time as \( \mathcal{N} \). Any parliament of any land, in any time is a subset \( N \in \mathcal{N} \). We denote by \( |N| \in \mathbb{N} \) the number of parties in the parliament \( N \), with \( \mathbb{N} \) the set of natural numbers, and each parliamentary party has a positive number of seats in this parliament. For a fixed \( |N| \), the set of all seats distributions is \( \mathbb{R}^{|N|}_0^+ \), with typical element \( (s_i)_{i \in N} \), where \( \mathbb{R}^{|N|}_0^+ \) is the positive orthant of the \( |N| \)-fold Euclidian space. That is, for any \( s \in \mathbb{R}^{|N|}_0^+ \), \( s_i > 0 \) is the number of seats of party \( i \). For an \( |N| \)-party parliament, a Power Index \( \beta \) is a real valued function defined on \( s \in \mathbb{R}^{|N|}_0^+ \) and \( q \in \mathcal{P} \):

\[
\beta : \mathcal{P} \times \mathbb{R}^{|N|}_0^+ \rightarrow \mathbb{R}^{|N|}_0^+
\]

We denote by \( p \in \mathbb{R}^{|N|}_0^+ \) the power vector with typical element \( p_i = \beta_i(s) \), \( i \in N \). We have \( p_i \geq 0 \) for each \( i \in N \) with at least one \( j \in N \) such that \( p_j > 0 \). For a given \( |N| \in \mathbb{N} \), the set of possible power distributions is \( \mathcal{P}^{|N|} \). The set of all possible power distributions is \( \mathcal{P} = \bigcup_{|N| \in \mathbb{N}} \mathbb{R}^{|N|}_0^+ \).

Any function \( E : \mathcal{P} \rightarrow \mathbb{R}^+_0 \) is denoted \( E^{|N|} \) when its domain of definition is \( \mathcal{P}^{|N|} \).

**Definition 5.** A Fragmentation Index is a function \( E \) from the set of possible power distributions to the set of positive real numbers:

\[
E : \mathcal{P} \rightarrow \mathbb{R}^+_0
\]

Finally we denote \( \hat{p} \) a permutation of \( p \) such that \( \hat{p}_1 \geq \hat{p}_2 \geq \cdots \geq \hat{p}_{|N|} \). The number of parties that are not dummies is \( |\hat{N}| \) and \( \hat{N} \subseteq N \) is the set of non-dummy parties. We turn now on properties that could be seen as natural or important for any equivalent-number fragmentation measure.

4.1 The Axioms

**Axiom 1. SYMmERTY (SYM)**

For all \( |N| \in \mathbb{N} \), for all \( p \in \mathcal{P}^{|N|} \), a fragmentation index \( E : \mathcal{P} \rightarrow \mathbb{R}^+_0 \) satisfies SYM if:

\[
E^{|N|}(p) = E^{|N|}(\pi p)
\]
where $\pi$ is any permutation matrix of conformable size. This axiom of symmetry simply says that two parliaments have equivalent fragmentation whenever they have the same number of parties and the same power distribution but differently assigned to parties. What matters only is the distribution of powers. Note also that the parties need not to be same in both parliaments, we just only need to have similar distributions of power numbers.

**Example 1.**

$E^5(0, 10, 10, 0, 10) = E^5(10, 10, 10, 0, 0)$

An important implication of $SYM$ is that fragmentation can be defined on ordered distributions:

$$E^{|N|}(p) = E^{|N|}(\bar{p})$$

**Axiom 2. DICTATOR(D)**

For all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{|N|}$, a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ satisfies D, with $p_i > 0$ for one and only one $i \in N$ and $p_j = 0$ for all $j \in N$, $j \neq i$, if we have:

$$E^{|N|}(p) = 1 \quad (4.2)$$

**Example 2.**

$E^5(10, 0, 0, 0, 0) = 1$

When one party can implement any decision by its own, its power measure must be the only strictly positive one in the distribution of power in the parliament, as imposed by consistent requirements for a power index. In such a case, fragmentation of power must take its lowest value, only one party being relevant, and the value should be 1.

**Axiom 3. NORMALIZATION (NOR)**

For all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{|N|}$, for any scalar $\alpha > 0$, a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ satisfies NOR if:

$$E^{|N|}(\alpha 1^{|N|}) = |N| \quad (4.3)$$

$$1^{|N|} = \underbrace{(1, \ldots, 1)}_{|N| \text{ times}}$$

When all parliamentary parties have the same power, the equivalent number is equal to the raw number of parties.

**Example 3.**

$E^3(10, 10, 10) = 3$
Axiom 4. UPPER BOUND (UB)

For all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{[N]}$, a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ satisfies UB if:

$$E^{[N]}(p) \leq |N| \tag{4.4}$$

An effective number of parties should not take a value higher than the raw number of parliamentary parties, which by NOR is the value attained when all parties have the same power. When $E^{[N]}(p) < |N|$, it indicates that some parties are more powerful than some others, and hence are more relevant.

Axiom 5. HOMOGENEITY (HO)

For all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{[N]}$, for any scalar $\alpha > 0$, a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ satisfies HO:

$$E^{[N]}(\alpha p) = \alpha E^{[N]}(p) = E^{[N]}(p) \tag{4.5}$$

The axiom HO says that a fragmentation measure should be homogeneous of degree zero. The unit of measurement has no impact on the degree of fragmentation of the distribution: rescaling two distributions do not alter relative evaluation of the two power distributions.

Example 4.

$$E^5(10, 10, 0, 10, 0) = (1/30)E^5(10 \times 1/30, 10 \times 1/30, 0 \times 1/30, 10 \times 1/30, 0 \times 1/30)$$

$$= E^5(1/3, 1/3, 0, 1/3, 0)$$

We can thus work with power shares $\bar{p}$.

Axiom 6. REPlication PRINCIPLE (RP)

For all $|M|, |N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{[N]}$, a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ satisfies RP if:

$$E^{[M||N]}(p') = |M| \times E^{[N]}(p) \tag{4.6}$$

$$p' = \underbrace{(p, \ldots, p)}_{|M| \text{ times}}$$

an $|M|$-fold replication of $p$.

When two power distributions are considered as equivalent by a fragmentation measure, their exact replication must also be ranked equivalent as well.

Example 5.

$$2 \times E^5(10, 10, 0, 10, 0) = E^{10}(10, 10, 0, 10, 0, 10, 0, 10, 0, 10)$$
This axiom will be convenient for the interpretation of rational fragmentation measures, *e.g.* of 1.5, because we know that replication of this distribution once makes it equivalent to a distribution of fragmentation 3.

**Axiom 7. DUMMY VOTER (DV)**

For all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{[N]}$, a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ satisfies DV if:

$$E^{(|N|+1)}(p, 0) = E^{[N]}(p) \quad (4.7)$$

Two assemblies with power distributions that only differ in the presence of dummy parties in one, must have the same degree of fragmentation. This is a key property that allows us to compare fragmentation with inequality measures, and that makes the difference between a fragmentation measure applied on seats rather on powers distribution.

**Example 6.**

$$E^5(10, 10, 0, 10, 0) = E^3(10, 10, 10)$$

**Axiom 8. CONTINUITY (CON)**

For all $|N| \in \mathbb{N}$, for every sequence $[p^m] \in \mathcal{P}^{[N]}$ converging to $p \in \mathcal{P}^{[N]}$, a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ is continuous if we have:

$$E^{[N]}([p^m]) \text{ converging to } E^{[N]}(p).$$

This property combined with the DV axiom, ensure that the value of a fragmentation index is not too sensitive to the presence or absence of parties with poor relevance.

**Axiom 9. POWER TRANSFER PRINCIPLE (PTP)**

For all $|N| \in \mathbb{N}$, for all $p, p' \in \mathcal{P}^{[N]}$ such that

$$|p'_i - p'_j| < |p_i - p_j|$$

and

$$p'_i + p'_j = p_i + p_j$$

and

$$p'_k = p_k$$

for all $k \neq i, j$,

a fragmentation index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ satisfies PTP if:

$$E^{[N]}(p') > E^{[N]}(p) \quad (4.8)$$

This principle says that in the comparison of two identical distributions except in the distance between two parties that is lower, fragmentation should increase.
Example 7. 
\[ E^5(10, 10, 10, 0, 0) < E^5(10, 10, 9, 1, 0) \]

Indeed this axiom is quite standard in the concentration literature. If one accepts it as natural or necessary, fragmentation as number equivalent is simply the inverse of an industrial concentration measure, and it has many implications on the possible measures and the value they take.

4.2 Implications of the axioms

Some interesting implications can be observed in combining adequately some of the above-mentioned axioms for a number-equivalent index.

Proposition 1. 
For all \(|N| \in \mathbb{N}\), for all \( p \in \mathcal{P}^{[N]} \), for any scalar \( \alpha > 0 \): if a fragmentation index \( E : \mathcal{P} \rightarrow \mathbb{R}_0^+ \) satisfies NOR + PTP + SYM, then it is bounded:

\[ E^{[N]}(p) \in [E^{[N]}(\alpha, 0, 0 \ldots); E^{[N]}((\alpha/|N|)1^{[N]})] \]

Proof. Let \( p \in \mathcal{P}^{[N]} \) be \((\frac{\alpha}{|N|}, \ldots, \frac{\alpha}{|N|})\), with \( \alpha \in \mathbb{R}_0^+ \). By NOR: \( E^{[N]}(p) = |N| \). Can we find a \( p' \in \mathcal{P}^{[N]} \) such that \( E^{[N]}(p') > E^{[N]}(p) \)?

By PTP we need to find a \( |p'_i - p'_j| < |p_i - p_j| \) with \( p'_i + p'_j = p_i + p_j \) and \( p'_k = p_k \) for all \( k \neq i, j \). But this is impossible since \( p_i = p_j \) for all \( i, j \in [1, \ldots, |N|] \). Thus \( p' \) is at best a permutation of \( p \). By SYM: \( E^{[N]}(p') = E^{[N]}(p) = |N| \) and this is the maximum value of \( E^{[N]} \).

Starting again from \( p \), any transfer from one party to another will thus diminish fragmentation by PTP. If \( \alpha \) is finite, after a finite number of transfers we end up with a \( p'' = (\alpha, 0, \ldots, 0) \), which is the distribution with minimum value of fragmentation, because all transfers diminishing fragmentation have been done. \( \square \)

Proposition 1 shows that the value taken by any fragmentation measure meeting these requirements takes its maximum value when all parties have the same power and its minimum value when there is a dictator.

Proposition 2. For all \(|N| \in \mathbb{N}\), for all \( p \in \mathcal{P}^{[N]} \), SYM, NOR, HO, DV, CON and PTP are independant: we can find fragmentation indices \( E : \mathcal{P} \rightarrow \mathbb{R}_0^+ \) satisfying all axioms but one.

Proof. see appendix A \( \square \)

Proposition 3. For all \(|N| \in \mathbb{N}\), for all \( p \in \mathcal{P}^{[N]} \), if a fragmentation index \( E : \mathcal{P} \rightarrow \mathbb{R}_0^+ \) satisfies SYM + PTP, then it is strictly S-Concave

Proof. see Encaoua and Jacquemin [10]. \( \square \)
**Definition 6. Self-Weighted Quasilinear Mean (SWQM)**

A fragmentation index \( E : \mathcal{P} \rightarrow \mathbb{R}_0^+ \), is called a self-weighted quasilinear mean if for all \(|N| \in \mathbb{N}\), for all \( p \in \mathcal{P}^{[N]} \), \( E^{[N]}(p) \) is of the form

\[
[E^{[N]}(p)]^{-1} = f^{-1} \left[ \sum_{i=1}^{N} \bar{p}_i f(\bar{p}_i) \right]
\]

where \( f : (0, 1] \rightarrow \mathbb{R}_0^+ \) is strictly monotonic and the function \( f^* : [0, 1] \rightarrow \mathbb{R}_0^+ \) defined

\[
f^*(x) = \begin{cases} 
xf(x) & , x \in (0, 1] \\
0 & , x = 0
\end{cases}
\]

is continuous on \([0, 1]\).

Note that power shares \( \bar{p} \) are used in the definition.

**Definition 7. Hannah & Kay Class of Fragmentation Indices**

For all \(|N| \in \mathbb{N}\), for all \( p \in \mathcal{P}^{[N]} \), a fragmentation index \( E : \mathcal{P} \rightarrow \mathbb{R}_0^+ \) belongs to the class of Hannah & Kay if it presents the following form:

\[
\left[ H_\alpha^{[N]}(p) \right]^{-1} = \left( \sum_{i=1}^{N} \bar{p}_i^\alpha \right)^{\frac{1}{\alpha - 1}}, \alpha > 0, \alpha \neq 1 = \prod_{i=1}^{N} \bar{p}_i^\alpha, \alpha = 1
\]

The Laakso-Taagepera Effective Number of Parties index (ENP) belongs to the Hannah & Kay class of indices when \( \alpha \) takes on the value of 2:

\[
ENP^{[N]}(p) = \frac{1}{\sum_{i=1}^{N} (\bar{p}_i)^2} \tag{4.9}
\]

**Theorem 4.** A self-weighted quasilinear form fragmentation index \( E : \mathcal{P} \rightarrow \mathbb{R}_0^+ \) is the Hannah & Kay index if and only if \( E \) satisfies the replication axiom.

**Proof.** see Chakravarty and Eichorn[5].

Hence, a fragmentation index of the SWQM form, with \( \alpha = 2 \) that respects the REP principle leads to the Laakso-Taagepera Effective Number of Parties index (equation 4.9). This full characterization will allow us to seek more efficiently to a better index, coping with the weaknesses of ENRP.
4.3 The Maximum Contribution Index of Fragmentation

If the Laakso & Taagepera Effective Number of Parties is widely applied in almost every recent comparative studies in politics, no mathematical characterization of the formula had never been proposed until now. As we have seen before, the derivation of the ENP comes directly from the Herfindhal index using seats shares instead of market or firms output shares in the calculation. Even for the case of the Herfindhal index, characterization has been presented far later its design. This way to proceed may explain why in some cases, the ENP or even its power counterpart ENRP, leads to unwanted results.

In this section we will define a new fragmentation index and justify it from the point of view that each party in a given parliament contributes to fragmentation proportionally to its relevance. Hence, the effect of each party on the fragmentation of a given parliament must be proportional to its relevance, which is measured by its power index in the parliament, and must also be a function of the distribution of power indices of all parties in the parliament. In order to capture this idea of marginal effect of a party on the fragmentation in a parliament, we introduce the next definition:

**Definition 8. Contribution to Fragmentation**

For an $|N|$-party parliament $N$, $|N| \in \mathbb{N}$, with a distribution of power indices $(p_i)_{i \in N} \in \mathbb{R}_{+}^{|N|}$, the contribution to fragmentation of a party $i \in N$ is a nonnegative strictly monotone and continuous function $cf$ of its power $p_i$:

$$cf(p_i) : \mathbb{R}_{+} \rightarrow C$$

with $p_i \in \mathbb{R}_{+}$ the power index of party $i$ in the parliament $N$ and $C$ the range of the values of the contribution to fragmentation.

Given a distribution of power indices in a parliament, each party contributes its own way to the fragmentation index. A fragmentation index is thus an aggregation of the contributions to fragmentation of every parliamentary party. Consequently, we define a fragmentation index as:

**Definition 9.** For all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}|N|$, a Fragmentation Index $E : \mathcal{P} \rightarrow \mathbb{R}_0^+$ is

$$E^{[N]}(p) = \sum_{i \in N} cf(p_i)$$

where $cf(p_i)$ is the contribution to fragmentation of party $i$ for the distribution of power $p$ in the parliament $N$.

The interpretation of such an index is straightforward: the fragmentation of the parliament under study is equivalent to a situation where we have the number calculated under (4.11) of parties with exactly the same...
degree of relevance.
As we said before, the contribution to fragmentation of a party in a given parliament is a way of asserting its relevance and is therefore proportional to its power index in the parliament. We introduce the following assumption under which the function $cf$ takes into account this proportionality of contribution to the power of a party inside a parliament with a given distribution of power:

**Definition 10.** For an $|N|$-party parliament $N$, $|N| \in \mathbb{N}$, with a distribution of power indices $(p_i)_{i \in N} \in \mathbb{R}_+^{|N|}$, the contribution to fragmentation of a party $i \in N$ is proportional to its power $p_i$ in the following sense:

$$cf(p_i) = cf\left(\frac{p_i}{2} + \frac{p_i}{2}\right) = cf\left(\frac{p_i}{2}\right) + cf\left(\frac{p_i}{2}\right) = 2cf\left(\frac{p_i}{2}\right)$$

(4.12)

for any $p_i \in \mathbb{R}_+$ with $cf : \mathbb{R}_+ \rightarrow \mathbb{R}$

Note that no restriction has been put yet on the range of the function $cf$.

A function $cf : \mathbb{R}_+ \rightarrow \mathbb{R}$ of the form (4.12) is called an additive function on $\mathbb{R}_+$. A function satisfying the functional equation (4.12) on the domain $\mathbb{R}_+$ is called a solution of the equation on that domain.

One solution to equation (4.12) for nonnegative variables, if $cf$ is assumed to be continuous, is given by

$$cf(p_i) = c.p_i$$

(4.13)

for all $p_i \in \mathbb{R}_+$ and $c$ a nonnegative real constant. Certainly, all functions of the form of (4.13) satisfy equation (4.12), whatever the constant $c$.

We look now at implications that axioms on a fragmentation index have on the equation (4.12) which must be identically satisfied for every value of the variables on the domain.

For any parliament $N$ with any number of parties $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{|N|}$, the fragmentation index (4.11) is

$$E^{|N|}(p) = \sum_{i \in N} cf(p_i)$$

By equation (4.12) we have

$$\sum_{i \in N} cf(p_i) = cf\left(\sum_{i \in N} p_i\right) = |N|.cf\left(\frac{\sum_{i \in N} p_i}{|N|}\right)$$
for all \( p_i \in \mathbb{R}^+ \).

A solution is given by

\[
c|N| \sum_{i \in N} \frac{p_i}{|N|} = c \sum_{i \in N} p_i
\]

for any positive constant \( c \).

By the axiom of \((\text{NOR})\), if we have a parliament \( N \) with distribution of power indices \( p \in \mathbb{R}_+^{|N|} \) such that \( p_j = \frac{\sum_{i \in N} p_i}{|N|} \) for all \( j \in N \), then

\[
E^{|N|}(p) = |N|
\]

Take now a parliament \( N \) where the number of non-dummy parties is \( \hat{N} \), with a distribution of power indices \( p \in \mathbb{R}_+^{|N|} \) such that every non-dummy party has the same power: \( p_i = k \) for all \( i \in \hat{N} \subseteq N \), \( k \in \mathbb{R}_+^+ \) and \( p_j = 0 \) for all \( j \in \left( N \setminus \hat{N} \right) \), by the axioms \( \text{NOR} \) and \( \text{DV} \):

\[
E^{|N|}(p) = |\hat{N}| = \sum_{i \in \hat{N}} cf(p_i) = \sum_{i \in \hat{N}} cf(k) + \sum_{j \in (N \setminus \hat{N})} cf(0)
\]

By equation (4.12) \( cf(0) = 0 \) and by \( \text{NOR} \): \( \sum_{i \in \hat{N}} cf(k) = |\hat{N}| \), which is consistent because only the relevant parties have a positive contribution. Suppose now that \( |\hat{N}| = 1 \), by the axiom of \((D)\)

\[
E^{|N|}(p) = cf(p_{\text{max}}) + \sum_{j \in (N \setminus \hat{N})} cf(0) = 1 = cf(p_{\text{max}})
\]

where \( p_{\text{max}} = \max_i p_i \) is here the power of the dictator.

Turning back to a parliament \( N \) with a power distribution \( p \in \mathbb{R}_+^{|N|} \) where \( p_i = k \) for all \( i \in \hat{N} \), \( \hat{N} > 1 \), \( k \in \mathbb{R}_0^+ \) and \( p_j = 0 \) for all \( j \in \left( N \setminus \hat{N} \right) \), we have

\[
\max_i p_i = k
\]

and it is the power of every non-dummy party. Then

\[
E^{|N|}(p) = \sum_{i \in \hat{N}} cf(p_{\text{max}}) = cf \left( \sum_{i \in \hat{N}} p_{\text{max}} \right) = cf \left( |\hat{N}| p_{\text{max}} \right) = |\hat{N}| cf(p_{\text{max}}) = |\hat{N}|
\]

therefore we must have \( cf(p_{\text{max}}) = 1 \). Moreover, \( \min_i cf(p_i) = 0 \) which is the contribution of a dummy party.
Definition 11. For an $|N|$-party parliament $N$, $|N| \in \mathbb{N}$, with a distribution of power indices $(p_i)_{i \in N} \in \mathbb{R}_+^{|N|}$, the contribution to fragmentation of a party $i \in N$ is a nonnegative strictly monotone and continuous function $cf$ of its power $p_i$:

$$cf(p_i) : \mathbb{R}_+ \to [0, 1]$$ \hspace{1cm} (4.14)

with $p_i \in \mathbb{R}_+$ the power index of party $i$ in the parliament $N$ and $[0, 1]$ is the interval of the values that can be taken by the function $cf$.

The last restriction on the range of $cf$ allows us to give a well-defined form for the index of fragmentation.

For all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{[N]}$, for all $p_i \in \mathbb{R}_+$, $cf(p_i) : \mathbb{R}_+ \to [0, 1]$ is bounded in the interval $[0, 1]$. We define

$$g(p_i) = cf(p_i) - p_i cf(1)$$

that also satisfies equation (4.12) and is bounded in $[0, 1]$, and for any arbitrary rational $r$, by appendix (B),

$$g(r) = 0$$

hence, $g(p_i + r) = g(p_i)$.

However, since it is possible to enter the interval $[0, 1]$ from each real $p_i$ by adding an appropriate rational number, $g(x)$ is bounded everywhere. If there exists a value $p_0$ such that $g(p_0) = A \neq 0$ where true, then $g(np_0) = n g(p_0) = nA$.

Hence, for sufficiently large $n$, function $g(p_i)$ could assume arbitrary large values, in contradiction to its boundedness demonstrated above.

Thus,

$$g(p_i) = 0$$

that is $cf(p_i) = p_i cf(1)$

We present this result in the theorem stated by Darboux(1880) and taken from Aczél [1].

Theorem 5. if for all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{[N]}$, the Cauchy’s functional equation

$$cf(p_i + p_j) = cf(p_i) + cf(p_j)$$

is satisfied for all $p_i, p_j \in \mathbb{R}_+$ and if the function $cf(p_i)$ is bounded in an interval, then

$$cf(p_i) = cp_i$$

for all nonnegative $p_i$ and a positive constant real $c$. 

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Lemma 6. if for all $|N| \in \mathbb{N}$, for all $p \in \mathcal{P}^{|N|}$, the Cauchy’s functional equation
\[ cf(p_i + p_j) = cf(p_i) + cf(p_j) \]
is satisfied for all $p_i, p_j \in \mathbb{R}^+$ and if the function $cf(p_i)$ is bounded in $[0, 1]$ and $cf(p_{max}) = 1$, then
\[ cf(p_i) = \frac{1}{p_{max}} p_i \]
for all $p_i \in \mathbb{R}^+$

Because
\[ cf(p_{max}) = p_{max} cf(1) = 1 \]
thus
\[ cf(1) = \frac{1}{p_{max}} \]

For any parliament $N$ with any number of parties $|N| \in \mathbb{N}$, for any power distribution $p \in \mathcal{P}^{|N|}$, a fragmentation index
\[ E^{|N|}(p) = \sum_{i \in N} cf(p_i) \]
with $cf(p_i)$ bounded on $[0, 1]$, satisfying the Cauchy’s equation and $cf(p_{max}) = 1$, then we define:

**Definition 12. Maximum Contribution Index of Fragmentation**
\[ E^{|N|}(p) = \sum_{i \in N} cf(p_i) = \sum_{i \in N} p_i = \frac{\sum_{i \in N} p_i}{p_{max}} \]

The Maximum Contribution to fragmentation index (MC) is the only index of the form (4.11) satisfying all the properties we have imposed on the function $cf$.

5 Properties of the MC Index of Fragmentation

First of all, it is easy to verify that MC index satisfies the axioms of $SYM$, $D$, $NOR$, $UB$, $HO$, $RP$, $DV$ and is continuous by $cf(.)$.

Secondly, the MC index is not an inequality index. Indeed, the generalized entropy class of inequality measures is
\[ GE(\alpha) = \frac{1}{\alpha^2 - \alpha} \left[ \frac{1}{|N|} \sum_{i=1}^{|N|} \left( \frac{y_i}{y} \right)^\alpha - 1 \right] \in [0, \infty] \quad (5.1) \]
where $y_i$ is the income of individual $i$, $|N| \in \mathbb{N}$ is the number of individuals and $\alpha$ is the weight given to distances between incomes of individuals. Cowell [7] shows that any measure of inequality is of the form of $GE$ if it satisfies the axioms of transfer Pigou-Dalton, income scale independence, anonymity, decomposability and principle of population. This last one is violated by MC. The principle of population (Dalton 1920) requires inequality measures to be invariant to replications of population: merging two identical distributions should not alter inequality: for any scalar $\lambda > 0 : I(x) = I(\lambda x)$ where $\lambda x$ is a concatenation of vector $x$, $\lambda$ times. Because MC satisfies the $RP$ axiom, it violates the principle of population and MC is thus not an inequality index.

Third, the MC index is not a concentration index either. There is no sense to transform market or output shares of firms into power distribution, and consequently the MC index fails to satisfy the Merger Principle. This property requires concentration to increase when we observe the fusion of two firms and the merging firm receives the total pre-merged combined market shares, all other firms’ market shares remaining the same. Suppose two parties agree to vote together on a series of propositions, but remain autonomous, each can split when it wants to. In that case, the combined power of both is just the sum of the power of each, remaining concentration unchanged, violating the Merger Principle. Suppose now two parties form a definitive cartel, merging their seats in the parliament, and acting as unitary actor. In this case, power has to be recalculated on the new distribution of seats, and the probability that the cartel will have less power then the sum of both elements is positive and known in the literature as the block paradox (see Felsenthal and Machover [11]), diminishing concentration and thus violating the Merger Principle.

Suppose now we apply nevertheless the MC index on market shares, and we observe a transfer of shares from bigger firm to a smaller one, with the bigger firm not the biggest one in the market (it is the second in size for example). If the Merger Principle is respected, we should observe a decrease in the concentration. But not with the MC index. Any transfer not concerning the bigger firm, unaltering the total of shares, will not change the value of the MC index.

Turning back to the properties of inequality indices, the counterpart of the Merger Principle is the Pigou-Dalton Transfer, which states that if a richer person transfers some income to a poorer one, inequality should diminish. Thus, if we apply the MC index and such a transfer is operated, if the richer one is not the first-ranked and that the total amount of income is the same, the MC index remains at the same level, violating the Pigou-Dalton Transfer principle.

\[ ^5 \text{For a complete list of the properties of concentration indices, we refer to Hannah and Kay [13].} \]
Hence, as we have defined the Maximum Contribution Index of Fragmentation it should not be applied to assess the inequality nor the concentration of a distribution. The MC index has been specifically developed to evaluate the extent to which power is scattered among the decision units in an assembly, and is thus applied on a distribution of power indices, representing the relevance of the actors. Besides its easiness to compute, it offers a straightforward and very intuitive interpretation in term of equivalent-number of relevant parties.

6 Conclusions

Measuring the fragmentation of a party-system with an effective-number index is a way to aggregate the parties’ seats such as to capture each one’s relevance in the decision-making process. Each party by its number of seats contribute to fragmentation. Up to now, no characterization had been presented of the well-used Effective Number of Parties, obtained by inversion of an industrial concentration measure. We provide it here for the political context.

Because relevance of a party is better represented by power indices rather than seats, because it does not convey any sense for a party to be more relevant than “one”, and because each party’s relevance is relatively evaluated, the contribution to fragmentation of the most relevant party should be one, squeezing the values of the contribution to fragmentation of each party between zero and one included. In such settings, the only satisfying measure is the Maximum Contribution Index of Fragmentation, which besides its easiness to compute has readily interpretation: the number it gives renders the actual situation equivalent to a hypothetical one where all parties have the same relevance. To conclude, our claim is that fragmentation measures have to be applied on power indices instead of seats, and once it has been done, any weighting would be superfluous, or even worsening.

A Appendix

We present here fragmentation indices satisfying 4 out of the axioms of SYM, HO, DV, CON, PTP in order to show independence of these 5 axioms.

\[
E^{\left|N\right|}(p) = \frac{1}{\sum_{i=1}^{n}(\bar{p}_i)^2} \begin{cases} 
\frac{1}{\bar{p}} & \text{if } \bar{p} = (1, 0, \ldots, 0) \\
0 & \text{otherwise}
\end{cases} \quad (A.1)
\]

Equation A.1 fails SYM.

\[
E^{\left|N\right|}(p) = \frac{1}{\sum_{i=1}^{n}(p_i)^2} \quad (A.2)
\]
Equation A.2 fails $HO$.

\[ E^{[N]}(p) = \frac{1}{\sum_{i=1}^{n}(\bar{p}_i)^2 - 1/n} \quad (A.3) \]

Equation A.3 fails $DV$.

\[ E^{[N]}(p) = \begin{cases} \frac{1}{\sum_{i=1}^{n}(\bar{p}_i)^2} & \text{if } \bar{p}_i > 0 \text{ for all } i \in N \\ \frac{1}{n \sum_{i=1}^{n}(\bar{p}_i)^2} & \text{if } s_i = 0 \text{ for some } i, \text{ where } \alpha > 0, \alpha \neq 1, \text{ is a constant.} \end{cases} \quad (A.4) \]

Equation A.4 fails $CON$.

\[ E^{[N]}(p) = \frac{1}{\sum_{i=1}^{n}(\bar{p}_i)} \quad (A.5) \]

Equation A.5 fails $PTP$, which completes the proof of independence. Note also that these 5 axioms are consistent: there exist functions that fulfil all the 5 axioms.

\[ E^{[N]}(p) = \frac{1}{\sum_{i=1}^{n}(\bar{p}_i)^2} \quad (A.6) \]

Equation A.6 satisfies $SYM$, $HO$, $DV$, $CON$, $PTP$, which proves consistency.

## B Appendix

The solution of the Cauchy’s equation is given in Aczél [1] pp.31-35 and we condense here some results we use in the text:

The Cauchy’s Basic Equation can be written as

\[ f(x + y) = f(x) + f(y) \quad (B.1) \]

or by induction,

\[ f(x_1 + x_2 + \cdots + x_n) = f(x_1) + f(x_2) + \cdots + f(x_n) \]

we pose \( x_k = x \) for \( k = 1, 2, \ldots, n \).

It follows directly that

\[ f(nx) = nf(x) \]

Thus, if \( x = (m/n)t \), then,

\[ nx = mt \]

and

\[ f(nx) = f(mt) \]

and also,

\[ nf(x) = mf(t) \]
that is,

\[ f \left( \frac{m}{n} t \right) = \frac{m}{n} f(t) \]  

(B.2)

If we let \( t = 1 \), \( f(1) = c \), then,

\[ f(x) = cx \]  

(B.3)

for every positive rational \( x \).

For \( x = 0 \), \( f(0) = 0 \) can be derived immediately from (B.1); thus (B.2) and (B.3) are also valid for \( m/n = 0 \) and \( x = 0 \), respectively.

For negative \( x \), we obtain by substituting \( y = -x \) in (B.1)

\[ f(x) = f(0) - f(-x) = -f(-x) \]

and thus (B.2) implies \( f(rt) = rf(t) \), for all real \( t \) and all rational \( r \).

if \( t = 1 \),

\[ f(r) = cr \]

for all rational \( r \).

Finally, we note that if \( f(x) \) is continuous everywhere, then it follows by taking limits on both sides of (B.3) that

\[ f(x) = cx \]

holds for all real \( x \).

References


